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Supertubes in bubbling backgrounds: Born-Infeld meets supergravity

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Abstract: We discuss two ways in which one can study two-charge supertubes as components of generic three-charge, three-dipole charge supergravity solutions. The first is using the Born-Infeld action of the supertubes, and the second is via the complete supergravity solution. Even though the Born-Infeld description is only a probe approximation, we find that it gives exactly the same essential physics as the complete supergravity solution. Since supertubes can depend on arbitrary functions, our analysis strengthens the evidence for the existence of three-charge black-hole microstate geometries that depend on an infinite set of parameters, and sets the stage for the computation of the entropy of these backgrounds. We examine numerous other aspects of supertubes in three-charge, three-dipole charge supergravity backgrounds, including chronology protection during mergers, the contribution of supertubes to the charges and angular momenta, and the enhancement of their entropy. In particular, we find that entropy enhancement affects supertube fluctuations both along the internal and the spacetime directions, and we prove that the charges that give the enhanced entropy can be much larger than the asymptotic charges of the solution. We also re-examine the embedding of five-dimensional black rings in Taub-NUT, and show that in different coordinate patches a ring can correspond to different four-dimensional black holes. Last, but not least, we show that all the three-charge black hole microstate geometries constructed so far can be embedded in $AdS_3 \times S^3$, and hence can be related to states of the D1-D5 CFT.

Keywords: Black Holes in String Theory, D-branes, AdS-CFT Correspondence

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JHEP07(2009)106

Contents

1 Introduction 1

2 Review of three-charge solutions 5
- 2.1 Three-charge solutions in the M2-M2-M2 (M-theory) frame 5
- 2.2 Three-charge solutions in the D0-D4-F1 duality frame 8
- 2.3 Three-charge solutions in the D1-D5-P duality frame 9

3 Regularity of supertubes in supergravity 9
- 3.1 Constraints from supertube regularity 9
- 3.2 Supertube regularity and spectral flow 11

4 Supertube probes and mergers in BPS solutions 12
- 4.1 Supertubes in a three-charge black hole background 12
- 4.2 Supertubes in a black-ring background 15
  - 4.2.1 The black-ring solution 15
  - 4.2.2 The black ring as a solution with a Gibbons-Hawking base 17
  - 4.2.3 Probing the black ring with two-charge supertubes 18
- 4.3 The black ring background: comparing the DBI analysis with supergravity 21
- 4.4 Black rings and three-charge two-dipole-charge supertubes 24
- 4.5 Supertubes in a general solution with a Gibbons-Hawking base 25
- 4.6 Gibbons-Hawking backgrounds: comparing the DBI analysis with supergravity 26
- 4.7 A comment on black rings in Taub-NUT and their four-dimensional charges 27

5 Chronology protection 28
- 5.1 Mergers of black rings with two-charge supertubes 28
- 5.2 Mergers of black rings with three-charge two-dipole-charge supertubes 30

6 Fluctuating supertubes and entropy enhancement 32
- 6.1 Flat space 32
- 6.2 The three-charge black hole 35
- 6.3 The three-charge black ring background 36
- 6.4 Solution with a general Gibbons-Hawking base 37
- 6.5 Comments on the supertube effective charges 38

7 Conclusions 39

A Three charge solutions and T-duality 42
- A.1 T-duality transformations 42
- A.2 Three charge solutions in different duality frames 43

B BPS solutions in D1-D5-P frame and their decoupling limit 48

C The angular momentum of the supertube 49

D Units and conventions 52

1 Introduction

The physics of two-charge supertubes is an essential ingredient in understanding the microstates of the D1-D5 system. Indeed, supergravity solutions for two charge supertubes with D1 and D5 charges and KKM dipole charge are smooth in six dimensions and can have arbitrary shape. Hence, they have an infinite dimensional classical moduli space, which, upon quantization, gives the entropy one expects from counting at weak-coupling: $S = 2\pi\sqrt{2N_1 N_5}$ [1–7].

While this entropy is considerable, it is nowhere near the entropy of a black hole with three charges: $S = 2\pi\sqrt{N_1 N_5 N_P}$ [8]. Hence, if one's goal is to prove that in the regime of parameters where the classical black hole exists one can find a very large number of string/supergravity configurations that realize enough microstates to account for the Bekenstein-Hawking entropy of this black hole [9–12], the entropy coming from two-charge supertubes does not appear to be large enough.

However, it has recently been found that the humble two-charge supertube has more to it than meets the eye: In a scaling supergravity background with large magnetic dipole fluxes it can undergo *entropy enhancement* [13]. That is, if one uses the Born-Infeld action to compute the entropy of a probe two-charge supertube placed in a background with three charges and three dipole charges, one finds that such a supertube can have an entropy that is *much* larger than that of the same supertube in empty space. The magnetic dipole-dipole interactions between the supertube and the background can greatly increase the capacity of the supertube to store entropy. Hence, the interaction with the supergravity background can enhance (or decrease) the entropy coming from the fluctuating shape of a supertube.

As yet, the fully back-reacted solution corresponding to a supertube of arbitrary shape has not been constructed and so the entropy enhancement calculation has only been done in a probe approximation. Nevertheless, in the absence of the fully back-reacted solutions, one can still pose a very sharp question, whose answer can tilt the balance one way or another in the quest to understand whether the black hole is a thermodynamic description of a very large number of horizonless microstates: "*Do two-charge supertubes that are solutions of the Born-Infeld equations of motion correspond to smooth solutions of supergravity once the back-reaction is included?*"

If the answer to this question is *yes*, then all the supertube microstates that were counted in [13] give smooth microstate solutions of supergravity, valid in the same regime of parameters where the classical black hole exists. Since the Born-Infeld counting might give a macroscopic (black-hole-like) entropy, this would imply that the same entropy could come from smooth supergravity solutions. Our goal in this paper is to show that the Born-Infeld description of a supertube does indeed capture all the essential physics of the

JHEP07(2009)106

complete supergravity solution and argue that the corresponding supergravity solution will be smooth in the D1-D5 duality frame.

First, we establish that when one has *both* a Born-Infeld and a supergravity description of supertubes in a three-charge, three-dipole-charge background, the two descriptions agree to the last detail. As we will see, this agreement can be rather subtle. For example, a supertube that is merging with a black ring appears to merge at an angle that depends on its charges but when this merger is described in supergravity, the merger appears to be angle-independent. The resolution of this rests upon the correct identification of constituent charges and the fact that such charges can depend upon "large" gauge transformations.

Another important fact we establish is that the solutions of the Born-Infeld action are always such that the corresponding solutions of supergravity are smooth in the duality frame where the supertube has D1 and D5 charges. Indeed, upon carefully relating the Born-Infeld and the supergravity charges, we will find that the equations that insure that a supertube is a solution of the Born-Infeld action are identical to the equations that insure that the corresponding supergravity solution is smooth.

One could take the position that our analysis here only implies the smoothness of round supertubes, which have both Born-Infeld and supergravity descriptions. It is possible that the wiggly supertubes (which, upon entropy enhancement, might give a black-hole-like entropy) could give rise to singular solutions when brought to the supergravity regime. While such a possibility cannot be fully excluded before the construction of the fully back-reacted wiggly supertubes, we have some rather strong reasons to believe it is highly unlikely. Indeed, if one investigates the conditions for smoothness of the supergravity solution and compares them to the Born-Infeld conditions, one finds that both the supergravity conditions and the Born-Infeld conditions are *local*. Hence, since any curve can be locally approximated as flat, our analysis indicates that no local properties of wiggly supertubes (like the absence of regions of high curvature) will differ from the local properties of round or flat supertubes. Thus one has a very reasonable expectation that supertubes of arbitrary shape will source smooth supergravity solutions.

In particular, if one considers supertubes of arbitrary shape in flat space, the solutions of the Born-Infeld action always give smooth supergravity solutions [4, 5]. If one now considers a three-charge, three-dipole charge solution containing supertubes whose wiggling scale is much smaller than the variation scale of the gauge fields of the background, one can perform a gauge transformation that locally removes the gauge fields and transforms a portion of this supertube into a portion with many wiggles of a supertube in flat space. Since the latter supertube is smooth, and since gauge transformations do not affect the smoothness of solutions, this implies that the original wiggly supertube is also giving a smooth solution.

Obviously the foregoing conclusion is restricted to the domain of validity of supergravity. If a supertube of arbitrary shape is very choppy, the local curvature will be roughly proportional to the inverse of the scale of the choppiness, and hence if the choppiness is Planck-sized then the curvature of the solution will also be Planck-sized. Such solutions are thus outside the domain of validity of supergravity. The main conclusion of our analysis is that supertubes whose wiggles are *not* Planck-sized will give smooth, low-curvature supergravity solutions.

JHEP07(2009)106

Our analysis does not establish whether the typical microstates of a certain black hole will have high curvature or will be well described in supergravity. However, it does establish that if the wiggles of the Born-Infeld supertubes that gave the typical microstates are not Planck-sized, the corresponding supergravity solutions will not be either.

The second aim of this paper is to clarify several issues related to embedding of black rings in Taub-NUT, and to the relation between the electric charges of the ring and those of the corresponding four-dimensional black holes. We show that when embedding a black ring solution in Taub-NUT one needs to use at least two coordinate patches. From the perspective of one patch, the electric charges are the ones found in [14], and the ring "angular" momentum along the Taub-NUT fiber (corresponding to the D0 charge in four dimensions) is given by the difference of the two five-dimensional angular momenta. The entropy is given by the $E_{7(7)}$ quartic invariant of these charges [15], as common for four-dimensional BPS black holes [16].

From the perspective of the other patch, the charges and the Kaluza-Klein angular momentum of the corresponding four-dimensional black hole are shifted, to certain values that have no obvious five-dimensional interpretation.[1] The entropy of the black ring is again given by the $E_{7(7)}$ quartic invariant, but now as a function of the shifted charges. The two four-dimensional black holes corresponding to the black ring are related by a gauge transformation, which shifts the Dirac string in the gauge potentials from one side of the ring to another.[2]

A third result in this paper is to verify chronology protection when supertubes and black rings are merged. While chronology protection is expected to be valid for this merger, the way it works is subtle. We compute the merger condition between a supertube and a black ring, and find that this condition depends on the position on the S^2 of the black ring where the supertube merges. We also find that neither very large nor very small supertubes can merge with the ring, for obvious reasons. If one varies the charge of the supertubes we find that mergers happen when the charge lies in a certain interval: At one extreme the supertube barely merges on the exterior of the ring while at the other it barely merges on the interior of the ring.

We also discuss a subtlety in identifying the constituent charges carried into the black ring by a merging supertube. We find that when the S^1 of the supertube curves around the S^2 of the black ring horizon, the charge brought in by a given supertube *must* depend on the S^2 azimuthal angle at which the supertube merges with the ring. Otherwise chronology is not protected. It would be most interesting to see how this comes about in the full supergravity merger solution.

The fourth aim of this paper is to present in detail, and to extend, the entropy enhancement calculation of [13]. Our analysis establishes that supertube entropy enhancement can come from supertube oscillation modes in *both* the internal space of the solution (T^4 in

[1]The asymptotic five-dimensional electric charge is the average between the four-dimensional electric charges in the two patches.

[2]Note that we can also perform a gauge transformation that shifts the four-dimensional electric charges to the asymptotic five-dimensional charges of the black ring [17]. The corresponding four-dimensional solution has two Dirac strings in the gauge potentials

our calculations) and from oscillations of supertubes in the transverse spacetime directions. We analyze entropy enhancement in black-ring backgrounds, in which the detailed computation is more straightforward than in generic solutions with a Gibbons-Hawking base. We find that, despite the presence of different (large) factors in the mode expansions, the fluctuations in the plane transverse to the ring give a contribution to the entropy that is identical to that coming from the fluctuations along the compactification torus.

If, as we expect, the enhanced entropy coming from these fluctuations will be black-hole-like, and therefore the fluctuating supertubes will give the typical microstates of the corresponding black hole, our analysis establishes that these microstates will have a non-trivial transverse size. We believe it important to calculate the amount of entropy enhancement coming from *all* the oscillations of the supertube. If the other transverse oscillations are more entropic than the torus ones, this would suggest that five-dimensional supergravity may be enough to capture the typical states of the black hole. On the other hand, if the torus and the transverse fluctuations are equally entropic (as hinted by our partial analysis), the typical states will probably have a curvature set by the compactification scale. Even if this scale is at the Planck scale, the microstate geometries constructed in supergravity will give a pretty good approximation of the rough features of the typical states (like the size, the density profile, the multipole moments). Hence the smooth microstate geometries will act as *representatives* of the typical black hole microstates [13, 18].

We begin in section 2 by presenting the general three-charge three-dipole-charge solutions in various duality frames that will be used throughout the paper. In particular, we give these solutions in the type IIA frame where the three charges correspond to D0 branes, D4 branes and F1 strings (the D0-D4-F1 frame), and in the type IIB duality frame where the three charges correspond to D1 branes, D5 branes and momentum (the D1-D5-P frame). We also obtain in these frames (for the first time to our knowledge) the exact form of the RR potentials when the base of the solution is a Gibbons-Hawking metric.

In section 3 we explore the regularity of the supergravity solutions corresponding to two-charge supertubes with D1 and D5 charges placed in three-charge three-dipole charge solutions. We find two *local* conditions that insure the absence of singularities near the supertube profile.

In section 4 we study probe two-charge supertubes in general three-charge solutions: black holes, black rings, and bubbling solutions with a Gibbons-Hawking base. We present a detailed analysis of two-charge and three-charge supertube probes in the background of a supersymmetric three-charge black ring. We also relate the supergravity and Born-Infeld charges of supertubes, and show that the supergravity smoothness conditions derived in section 3 agree with the ones derived from the Born-Infeld action. In section 5 we study mergers of the supertube with the black ring and discuss chronology protection and black hole thermodynamics during these mergers.

Section 6 contains an in-depth derivation of the entropy coming from oscillations of supertubes, illustrating the entropy enhancement mechanism presented in [13] for black rings, and general solutions with a Gibbons-Hawking base. Section 7 is devoted to conclusions.

In appendix A we give the details of the T-duality transformations of three-charge three-dipole charge solutions in various duality frames. We also show how to compute the

JHEP07(2009)106

RR potentials corresponding to these solutions in various duality frames. In appendix B we take a decoupling limit for general three-charge three-dipole charge solutions in D1-D5-P frame, which leads to an asymptotically $AdS_3 \times S^3 \times T^4$ geometry. This establishes that all the black hole and black ring microstate solutions constructed so far are dual to states of the D1-D5 CFT, and serves as a starting point for analyzing these microstates using holographic anatomy in the context of the AdS_3/CFT_2 correspondence [19]. In appendix C we compute the angular momentum of a supertube in several three-charge backgrounds and in appendix D we give the units and conventions used throughout our calculations.

2 Review of three-charge solutions

2.1 Three-charge solutions in the M2-M2-M2 (M-theory) frame

Three-charge solutions with four supercharges are most simply written in the M-theory duality frame in which the three charges are treated most symmetrically and correspond to three types of M2 branes wrapping three T^2's inside T^6 [20]. The metric is:

$$\begin{aligned} ds_{11}^2 = & -(Z_1Z_2Z_3)^{-\frac{2}{3}}(dt+k)^2 + (Z_1Z_2Z_3)^{\frac{1}{3}}\, ds_4^2 \\ & + \left(Z_2Z_3Z_1^{-2}\right)^{\frac{1}{3}}(dx_5^2+dx_6^2) + \left(Z_1Z_3Z_2^{-2}\right)^{\frac{1}{3}}(dx_7^2+dx_8^2) + \left(Z_1Z_2Z_3^{-2}\right)^{\frac{1}{3}}(dx_9^2+dx_{10}^2), \end{aligned} \tag{2.1}$$

where ds_4^2 is a four-dimensional hyper-Kähler metric [20–22].[3] The solution has a non-trivial three-form potential, sourced both by the M2 branes (electrically) and by the M5 dipole branes (magnetically):

$$\mathcal{A} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8 + A^{(3)} \wedge dx_9 \wedge dx_{10}. \tag{2.2}$$

The magnetic contributions can be separated from the electric ones by defining the "magnetic field strengths:"

$$\Theta^{(I)} \equiv dA^{(I)} + d\left(\frac{(dt+k)}{Z_I}\right), \qquad I = 1,2,3. \tag{2.3}$$

Finding supergravity solutions for this system then boils down to solving the following system of BPS equations:[4]

$$\begin{aligned} \Theta^{(I)} &= \star_4 \Theta^{(I)}\,, \\ \nabla^2 Z_I &= \frac{1}{2} C_{IJK} \star_4 \left(\Theta^{(J)} \wedge \Theta^{(K)}\right), \\ dk + \star_4 dk &= Z_I \Theta^I\,. \end{aligned} \tag{2.4}$$

In these equations, $\star_4$ is the Hodge dual in the four-dimensional base space, ds_4^2, and $C_{IJK} = |\epsilon_{IJK}|$. If the four-dimensional base manifold has a triholomorphic U(1) isometry then the metric on the base can be put in a Gibbons-Hawking (GH) form [28, 29]:

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V\, d\vec{y} \cdot d\vec{y}\,, \tag{2.5}$$

[3]This metric can have regions of signature +4 and signature −4 [23–27], and for this reason we usually refer to it as ambipolar.

[4]These equations also give supersymmetric solutions when the T^6 is replaced by a Calabi-Yau three-fold, and C_{IJK} is replaced by the triple intersection numbers of this three-fold.

where V is a harmonic function on the $\mathbb{R}^3$ spanned by (y_1, y_2, y_3) and $\vec{\nabla} \times \vec{A} = \vec{\nabla} V$. For such metrics, the BPS equations (2.5) can be solved explicitly [14, 30]. The most general solution can be written in terms of eight harmonic functions (V, K^I, L_I, M) on the $\mathbb{R}^3$ base of the GH space.[5] It is convenient to introduce the vielbeins:

$$\hat{e}^1 = V^{-\frac{1}{2}}\,(d\psi + A)\,, \qquad \hat{e}^{a+1} = V^{\frac{1}{2}}\,dy^a\,, \quad a = 1,2,3\,, \tag{2.6}$$

then one has

$$\Theta^{(I)} = -\sum_{a=1}^{3} \left(\partial_a \left(V^{-1} K^I\right)\right) \left(\hat{e}^1 \wedge \hat{e}^{a+1} + \frac{1}{2}\,\epsilon_{abc}\,\hat{e}^{b+1} \wedge \hat{e}^{c+1}\right). \tag{2.7}$$

The three gauge fields, $A^{(I)}$, can be written as

$$A^{(I)} = B^{(I)} - \frac{1}{Z^I}(dt + k)\,, \tag{2.8}$$

where

$$B^{(I)} = V^{-1} K^I\,(d\psi + A) + \vec{\xi}^{(I)} \cdot d\vec{y}\,, \qquad \vec{\nabla} \times \vec{\xi}^{(I)} \equiv -\vec{\nabla} K^I\,. \tag{2.9}$$

The functions Z_I and the angular momentum one-form k are given by

$$Z_I = \frac{C_{IJK}}{2}\frac{K^J K^K}{V} + L_I\,, \qquad k = \mu(d\psi + A) + \vec{\omega} \cdot d\vec{y}\,, \tag{2.10}$$

where

$$\mu = \frac{C_{IJK}}{6}\frac{K^I K^J K^K}{V^2} + \frac{K^I L_I}{2V} + M \tag{2.11}$$

and $\vec{\omega}$ satisfies the equation

$$\vec{\nabla} \times \vec{\omega} = V \vec{\nabla} M - M \vec{\nabla} V + \frac{1}{2}\left(K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I\right). \tag{2.12}$$

This solution can describe five-dimensional black holes, circular black rings and supertubes, as well as smooth "bubbling solutions" and an arbitrary superposition of these objects. Upon compactifying to four dimensions, all these reduce to BPS multi-center black-hole configurations [35, 36] of the type first considered in [37].

The harmonic functions are usually chosen to be sourced by simple poles:

$$\begin{aligned} V &= \epsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}\,, \qquad & K^I &= \kappa_0^I + \sum_{j=1}^{N} \frac{k_j^I}{r_j}\,, \\ L_I &= l_0^I + \sum_{j=1}^{N} \frac{l_j^I}{r_j}\,, \qquad & M &= m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j}\,, \end{aligned} \tag{2.13}$$

where $r_j = |\vec{y} - \vec{y}_j|$ and N is the number of centers. We think of the residues of the poles of these functions as defining the GH charges of the corresponding solution. As was

[5] For M-theory compactifications on a generic Calabi-Yau three-fold the number of harmonic functions will be $2h^{1,1} + 2$. See [34] for a discussion of such solutions.

JHEP07(2009)106

discussed in [38], gauge transformations and spectral flow can reshuffle these charges, but this produces physically equivalent solutions.

A necessary (but not sufficient) condition for the solutions to be free of closed timelike curves (CTC's) is to satisfy the "integrability equations," or "bubble equations," [24, 25, 37]:

$$\sum_{j=1,j\neq i}^{N} \frac{\langle \hat{Q}_i|\hat{Q}_j\rangle}{r_{ij}} = 2(\varepsilon_0 m_i - m_0 q_i) + \sum_{I=1}^{3}\left(k_0^I l_i^I - l_0^I k_i^I\right) \tag{2.14}$$

where $\langle \hat{Q}_i|\hat{Q}_j\rangle$ is the symplectic product[6] between the eight-vectors of charges at the points i and j

$$\langle \hat{Q}_i|\hat{Q}_j\rangle \equiv 2(m_j q_i - q_j m_i) + \sum_{I=1}^{3}\left(l_j^I k_i^I - k_j^I l_i^I\right) \ . \tag{2.15}$$

For smooth solutions with multiple GH centers the parameters of the solution must also satisfy the additional regularity constraints:

$$l_j^I = -\frac{C_{IJK}}{2}\frac{k_j^J k_j^K}{q_j}\,, \qquad m_j = \frac{k_j^1 k_j^2 k_j^3}{2q_j^2}\,, \tag{2.16}$$

These are required to cancel the singularities in Z_I and μ and with these choices the integrability equations (2.14) reduce to the bubble equations considered in [24, 25].

One can arrange for the global absence of CTC's by requiring that there is a well-defined, global time function [25]. This is much more stringent than the bubble equations (which only eliminate CTC's in the neighborhood of the GH points) and means that the following inequality should be satisfied globally [24, 25]:

$$Z_1 Z_2 Z_3 V - \mu^2 V^2 - |\omega|^2 \geq 0\,. \tag{2.17}$$

This condition is very hard to check in general and usually has to be checked numerically for particular solutions.

As we mentioned earlier, in order to study two-charge supertubes in backgrounds like those presented here, it is useful to dualize to a frame in which the two-charge supertube action is simple. One such frame is where the three electric charges correspond to D0 branes, D4 branes and F1 strings and the supertube carries D0 and F1 electric charges and D2 dipole charge [1]. On the other hand, in order to study the supergravity solutions describing supertubes in black-ring or bubbling backgrounds, it is useful to work in a duality frame in which the supergravity solution for the supertubes is smooth. In this frame the electric charges of the background correspond to D1 branes, D5 branes, and momentum P, and the supertube carries D1 and D5 charges, with KKM dipole charge. We therefore dualize the foregoing M-theory solution to these frames and give all the details of the solutions explicitly.

[6] This product is sometimes called the Dirac-Schwinger-Zwanziger product.

JHEP07(2009)106

2.2 Three-charge solutions in the D0-D4-F1 duality frame

Here we will present the three-charge solutions in the duality frame in which they have electric charges corresponding to D0 branes, D4 branes, and F1 strings, and dipole charges corresponding to D6, D2 and NS5 branes. We use the T-duality rules (given in appendix A) to transform field-strengths. It should be emphasized that our results are correct for any three-charge solution (including those without a tri-holomorphic U(1) [41]), however, finding the explicit form of the RR and NS-NS *potentials* (which is crucial if we want to investigate this solution using probe supertubes) is straightforward only when the solution can be written in Gibbons-Hawking form.

Label the coordinates by $(x^0, \dots, x^8, z)$.[7] The electric charges N_1, N_2 and N_3 of the solution then correspond to:

$$N_1: \, D0 \qquad N_2: \, D4\,(5678) \qquad N_3: \, F1\,(z) \tag{2.18}$$

where the numbers in the parentheses refer to spatial directions wrapped by the branes and $z \equiv x^{10}$. The magnetic dipole moments of the solutions correspond to:

$$n_1: \, D6\,(y5678z) \qquad n_2: \, D2\,(yz) \qquad n_3: \, NS5\,(y5678)\,, \tag{2.19}$$

where y denotes the brane profile in the spatial base, $(x^1, \dots, x^4)$. The metric of the solution is:

$$ds^2_{IIA} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\, ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3} dz^2 + \sqrt{\frac{Z_1}{Z_2}}\left(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2\right)\,. \tag{2.20}$$

The dilaton and the Kalb-Ramond fields are:

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^3}{Z_2 Z_3^2}\right)\,, \qquad\qquad B = -dt\wedge dz - A^{(3)}\wedge dz\,. \tag{2.21}$$

The RR field strengths are

$$F^{(2)} = -\mathcal{F}^{(1)}\,, \qquad \widetilde{F}^{(4)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)})\wedge dz\,, \tag{2.22}$$

where we define $\mathcal{F}^{(I)} \equiv dA^{(I)}$ and $\star_5$ is the Hodge dual with respect to the five dimensional metric:

$$ds_5^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\, ds_4^2\,. \tag{2.23}$$

The foregoing results are valid for any three-charge solution with an arbitrary hyper-Kähler base. As we show in appendix A, when the base has a Gibbons-Hawking metric one can easily find the RR 3-form potential:

$$C^{(3)} = \left(\zeta_a + V^{-1}K^3\xi_a^{(1)}\right)\Omega_-^{(a)}\wedge dz - \left(Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\right)\wedge dz\,, \tag{2.24}$$

where $\xi_a^{(1)}$ and ζ_a are defined by equations (2.9) and (A.36). Thus we have the full three-charge supergravity solution in the D0-D4-F1 duality frame. In section 4 we will perform a probe analysis in this class of backgrounds using the DBI action for supertubes with D0 and F1 electric and D2 dipole charge.

[7] See appendix A for more details about the brane configuration that we use.

2.3 Three-charge solutions in the D1-D5-P duality frame

One can T-dualize the solution above along z to obtain a solution with D1, D5 and momentum charges:

$$N_1 : D1\,(z) \qquad N_2 : D5\,(5678z) \qquad N_3 : P\,(z) \tag{2.25}$$

and dipole moments corresponding to wrapped D1 branes, D5 branes and Kaluza Klein Monopoles (kkm):

$$n_1 : D5\,(y5678) \qquad n_2 : D1\,(y) \qquad n_3 : kkm\,(y5678z)\,. \tag{2.26}$$

The metric is

$$ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\left(dz + A^{(3)}\right)^2 \tag{2.27}$$

$$+\sqrt{\frac{Z_1}{Z_2}}\,\left(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2\right) \tag{2.28}$$

and the dilaton and the Kalb-Ramond field are:

$$\Phi = \frac{1}{2}\log\left(\frac{Z_1}{Z_2}\right), \qquad B = 0\,. \tag{2.29}$$

The only non-zero RR three-form field strength is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5\left(\mathcal{F}^{(2)}\right) - \mathcal{F}^{(1)}\wedge\left(dz - A^{(3)}\right)\,. \tag{2.30}$$

If we specialize our general result to the supersymmetric black ring solution in the D1-D5-P frame then it agrees (up to conventions) with [42]. It is also elementary to find the RR two-form potential for a general BPS solution with GH base in D1-D5-P frame. This can be done by T-dualizing the IIA D0-D4-F1 result (2.24), to obtain:

$$C^{(2)} = \left(\zeta_a + V^{-1}K^3\xi_a^{(1)}\right)\,\Omega_-^{(a)} - \left(Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\right) \tag{2.31}$$

$$+A^{(1)}\wedge(A^{(3)} - dz - dt) + dt\wedge(A^3 - dz)\,, \tag{2.32}$$

where again $\xi_a^{(1)}$ and ζ_a are defined in equations (2.9) and (A.36). This is the full three-charge supergravity solution in the D1-D5-P duality frame. As shown in [5], two-charge supertubes in flat space are regular only in this duality frame, so our general result can be used to analyze the regularity of two charge supertubes in a general three-charge solution. This will be the subject of the next section.

3 Regularity of supertubes in supergravity

3.1 Constraints from supertube regularity

Consider the D1-D5-P solutions in which one of the centers has vanishing GH charge, and non-trivial D1 and D5 electric charges. Generally such a solution is not regular and can have a horizon or a naked singularity. However, the solution will be regular if one arranges the charges at this point to be those of a two-charge supertube.

Suppose that at $r_1 = 0$ we have a round two-charge supertube with one dipole charge. We take the latter to be k_1^3 and so we have $k_1^1 = k_1^2 = 0$ and $l_1^3 = 0$. This means that in the neighborhood of a two-charge supertube at $r_1 = 0$, we must have:

$$Z_I \sim \mathcal{O}(r_1^{-1}) , \quad I = 1,2 ; \qquad V, Z_3 \sim \text{finite} . \tag{3.1}$$

The six-dimensional metric in IIB frame can be re-written as:

$$ds_6^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\,\left(dz + A^{(3)}\right)^2 . \tag{3.2}$$

To check regularity along the supertube one must examine potential singularities along the ψ-fiber by collecting all the $(d\psi + A)^2$ terms in (3.2):

$$(Z_1 Z_2)^{-\frac{1}{2}}\, V^{-2}\left[Z_3\,(K^3)^2 - 2\mu V K^3 + Z_1 Z_2 V\right]\,(d\psi + A)^2 . \tag{3.3}$$

For regularity as $r_1 \to 0$, one must have:

$$\lim_{r_1 \to 0}\, r_1^2\left[Z_3\,(K^3)^2 - 2\mu V K^3 + Z_1 Z_2 V\right] = 0 . \tag{3.4}$$

Next there is a potential problem with CTC's coming from Dirac strings in ω. For ω to have a Dirac string originating at $r_1 = 0$, the source terms in the equation for $\vec\omega$ must have a piece that behaves as a constant multiple of $\vec\nabla\frac{1}{r_1}$. To examine this, it is easier to use (2.12) and recall that Z_3, K^1, K^2 and V are finite as $r_1 \to 0$. Thus the only sources of "dangerous terms" are $V\vec\nabla\mu$ and $Z_3\vec\nabla K^3$. Since V and Z_3 are finite at $r_1 = 0$, there will be no Dirac strings starting at $r_1 = 0$ if and only if:

$$\lim_{r_1 \to 0}\, r_1\left[V\mu - Z_3\,K^3\right] = 0. \tag{3.5}$$

The two conditions, (3.4) and (3.5), guarantee that the supertube smoothly caps off the spatial geometry and are the generalization to three-charge three-dipole backgrounds of the conditions for smooth cap-off in [5].

One can massage these conditions using (3.5) to eliminate all the explicit K^3 terms in (3.4). The condition (3.4) may then be written as

$$\lim_{r_1 \to 0}\, r_1^2\,\mathcal{Q} = 0 . \tag{3.6}$$

where $\mathcal{Q}$ is the E_7 invariant that determines the four-dimensional horizon area [14, 15]:

$$\begin{aligned}\mathcal{Q} &\equiv Z_1 Z_2 Z_3 V - \mu^2 V^2 &(3.7)\\ &= -M^2 V^2 - \frac{1}{3}\,M\,C_{IJK}K^I\,K^J\,K^k - M\,V\,K^I\,L_I - \frac{1}{4}\,(K^I L_I)^2 &\\ &\quad + \frac{1}{6}V C^{IJK}L_I L_J L_K + \frac{1}{4}C^{IJK}C_{IMN}L_J L_K K^M K^N . &(3.8)\end{aligned}$$

We will therefore refer to (3.6) as the quartic constraint. Note that the right-hand side of (2.12) is the quadratic E_7 invariant, and so we may view (3.5) as the "quadratic constraint." It is, however, convenient to rewrite this constraint by eliminating μ from (3.4) using (3.5). One then obtains:

$$\lim_{r_1 \to 0} r_1^2 \left[V Z_1 Z_2 - Z_3 \, (K^3)^2 \right] = 0 \,. \tag{3.9}$$

We will use (3.5) and (3.9) as the independent constraints because they are simplest to apply.

In flat space the supertube solution has $V = \frac{1}{r}$, $K^1 = K^2 = 0$ and $Z_3 = 1$, and equation (3.9) determines the radius of the supertube in terms of its charges, and (3.5) fixes the parameter m_1 of (2.13), and thus determines the angular momentum of the supertube in terms of its radius and charges.

3.2 Supertube regularity and spectral flow

As explained in [38], one can obtain a solution with a supertube inside a general three-charge solution by spectrally flowing a smooth horizonless bubbling solution.[8] Since spectral flow is implemented by a coordinate change in six dimensions, it cannot affect the smoothness or the regularity of the solution. Equivalently, regularity is determined by placing conditions on quadratic and quartic E_7 invariants, and as shown in [38], these are invariant under spectral flow transformations.

We therefore expect that the equations that determine the smoothness of supertubes, (3.4), (3.5) and (3.9), should be related by spectral flow to the equations that determine the smoothness of a usual bubbling solution. Indeed, consider the spectral flow transformation (see [38] for more detail):

$$\widetilde{V} = V + \gamma \, K^3 \,, \qquad \widetilde{K}^1 = K^1 - \gamma \, L_2 \,, \qquad \widetilde{K}^2 = K^2 - \gamma \, L_1 \,, \qquad \widetilde{K}^3 = K^3 \tag{3.10}$$

$$\widetilde{L}_1 = L_1 \,, \qquad \widetilde{L}_2 = L_2 \,, \qquad \widetilde{L}_3 = L_3 - 2 \, \gamma \, M \,, \qquad \widetilde{M} = M \,, \tag{3.11}$$

with

$$\gamma = -\frac{q_1}{k_1^3} \,. \tag{3.12}$$

This transformation maps a GH bubbled solution to a GH bubbled solution with a supertube at $r_1 = 0$. Under this spectral flow one also has:

$$\widetilde{Z}_1 = \left(\frac{V}{\widetilde{V}} \right) Z_1 \,, \qquad \widetilde{Z}_2 = \left(\frac{V}{\widetilde{V}} \right) Z_2 \,, \qquad \tilde{\mu} = \left(\frac{V}{\widetilde{V}} \right) \left(\mu - \gamma \frac{Z_1 Z_2}{\widetilde{V}} \right) \,, \tag{3.13}$$

$$\widetilde{Z}_3 = \left(\frac{\widetilde{V}}{V} \right) Z_3 + \gamma^2 \left(\frac{Z_1 Z_2}{\widetilde{V}} \right) - 2 \, \gamma \mu \,. \tag{3.14}$$

[8] See also [39, 40].

In the usual bubbling solution, regularity requires that the Z_I are finite and $\mu \to 0$ as $r_1 \to 0$. In the solution with the supertube one can use this and (3.14) to verify that:

$$\lim_{r_1\to 0} r_1 \left[\widetilde{V}\tilde{\mu} - \widetilde{Z}_3\,\widetilde{K}^3\right] = -\gamma \lim_{r_1\to 0} r_1 \left(\frac{V Z_1 Z_2}{\widetilde{V}}\right)\left(1+\gamma\,\frac{K^3}{V}\right), \tag{3.15}$$

$$\lim_{r_1\to 0} r_1^2 \left[\widetilde{V}\widetilde{Z}_1\widetilde{Z}_2 - \widetilde{Z}_3\,(\widetilde{K}^3)^2\right] = \lim_{r_1\to 0} r_1^2 \left(\frac{Z_1 Z_2}{\widetilde{V}}\right)\left(V^2 - \gamma^2 (K^3)^2\right). \tag{3.16}$$

Both of these vanish by virtue of (3.12) and the finiteness of the Z_I and $\widetilde{V}$ as $r_1 \to 0$. Hence, the equations determining the smoothness and regularity of two-charge supertubes are related by spectral flow to those determining the smoothness and regularity of usual three charge bubbling solution.

4 Supertube probes and mergers in BPS solutions

We now turn to the description of supertubes in terms of the Dirac-Born-Infeld (DBI) action. Our purpose is four-fold: to show that the supertubes that are solutions of the DBI action back-react into smooth horizonless geometries; to identify the Born-Infeld charges of supertube with those of the corresponding solution with a Gibbons-Hawking base; to facilitate the analysis of chronology protection in section 5, and to set the stage for the entropy enhancement calculation in section 6.

We begin with a review of supertubes in the background of a three-charge rotating BPS (BMPV) black hole [43, 44], and then extend this to a black ring, and to more general three-charge backgrounds. The first goal is to show that the Born-Infeld calculation captures the same essential data that is given by the regularity conditions of the fully back-reacted supergravity solution. We will also show that the Born-Infeld analysis and exact supergravity analysis give the same merger conditions for supertubes with black rings.

4.1 Supertubes in a three-charge black hole background

As a warm up exercise, we first consider a probe supertube with two charges and one dipole charge in the background of a three-charge (BMPV) black hole. This example was considered in [43, 44] and was generalized to a probe supertube with three charges and two dipole charges in [45]. The full supergravity solution describing a BMPV black hole on the symmetry axis of a black ring with three charges and three dipole charges was found in [20, 30], and a more general solution in which the black hole is not at the center of the ring was found in [46]

First, we need the BMPV black hole solution in the D0-D4-F1 duality frame. The metric (in the string frame) is:

$$\begin{aligned} ds_{10}^2 = &-\frac{1}{\sqrt{Z_1 Z_2} Z_3}(dt+k)^2 + \sqrt{Z_1 Z_2}\,\left(d\rho^2 + \rho^2\left(d\vartheta^2 + \sin^2\vartheta\varphi_1^2 + \cos^2\vartheta d\varphi_2^2\right)\right) \\ &+\frac{\sqrt{Z_1 Z_2}}{Z_3}\,dz^2 + \sqrt{\frac{Z_1}{Z_2}}\,ds_{T^4}^2 \end{aligned} \tag{4.1}$$

and the dilaton and the Kalb-Ramond field are given by:

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^3}{Z_2 Z_3^2}\right), \qquad B = (Z_3^{-1} - 1)dt \wedge dz + Z_3^{-1} k \wedge dz\,. \tag{4.2}$$

The non-trivial RR potentials are:

$$C^{(1)} = (Z_1^{-1} - 1)dt + Z_1^{-1}k\,, \qquad C^{(3)} = -(Z_2 - 1)\rho^2\cos^2\vartheta d\varphi_1 \wedge d\varphi_2 \wedge dz + Z_3^{-1} dt \wedge k \wedge dz\,. \tag{4.3}$$

The one-form k and the functions Z_I are given by

$$k = k_1 d\varphi_1 + k_2 d\varphi_2 = \frac{J}{\rho^2}(\sin^2\vartheta d\varphi_1 - \cos^2\vartheta d\varphi_2)\,, \qquad Z_I = 1 + \frac{Q_I}{\rho^2}\,, \tag{4.4}$$

where J is the angular momentum of the black hole. The charges, Q_1, Q_2 and Q_3 correspond to the respective D0 brane, D4 brane and F1 string charges of the black hole.

This solution is indeed a BPS, five-dimensional, rotating black hole [47] with an event horizon at $r = 0$, whose area is proportional to $\sqrt{Q_1 Q_2 Q_3 - J^2}$. For $J^2 > Q_1 Q_2 Q_3$ the solution has closed time-like curves and is unphysical.

We will denote the world-volume coordinates on the supertube by ξ^0, ξ^1 and $\xi^2 \equiv \theta$. To make the supertube wrap z we take $\xi^1 = z$ and we will fix a gauge in which $\xi^0 = t$. Note that $z \in (0, 2\pi L_z)$. The profile of the tube, parameterized by θ, lies in the four-dimensional non-compact $\mathbb{R}^4$ parameterized by $(\rho, \vartheta, \varphi_1, \varphi_2)$ and for a generic profile all four of these coordinates will depend on θ.

It is convenient to use polar coordinate (u, φ_1) and (v, φ_2) in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, where the $\mathbb{R}^4$ metric takes the form:

$$ds_4^2 = d\rho^2 + \rho^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2\right) = du^2 + u^2 d\varphi_1^2 + dv^2 + v^2 d\varphi_2^2\,. \tag{4.5}$$

There is also a gauge field, $\mathcal{F}$, on the world-volume of the supertube. Supersymmetry requires that $\mathcal{F}$ essentially has constant components and we can then boost the frames so that $\mathcal{F}_{t\theta} = 0$.

In this frame supersymmetry also requires $\mathcal{F}_{tz} = 1$ [1]. For the present we take

$$2\pi\alpha' F \equiv \mathcal{F} = \mathcal{F}_{tz} dt \wedge dz + \mathcal{F}_{z\theta} dz \wedge d\theta\,, \tag{4.6}$$

where the components are constant. Keeping $\mathcal{F}_{tz}$ as a variable will enable us to extract the charges below.

The supertube action is a sum of the DBI and Wess-Zumino (WZ)actions:

$$S = -T_{D2}\int d^3\xi e^{-\Phi}\sqrt{-\det\left(\widetilde{G}_{ab} + \widetilde{B}_{ab} + \mathcal{F}_{ab}\right)} + T_{D2}\int d^3\xi\left[\widetilde{C}^{(3)} + \widetilde{C}^{(1)} \wedge (\mathcal{F} + \widetilde{B})\right]\,, \tag{4.7}$$

where, as usual, $\widetilde{G}_{ab}$ and $\widetilde{B}_{ab}$ are the induced metric and Kalb-Ramond field. We have also chosen the orientation such that $\epsilon_{tz\theta} = 1$. It is also convenient to define the following induced quantities on the world-volume:

$$\Delta_{\mu\nu} = \partial_\mu u \partial_\nu u + u^2 \partial_\mu\varphi_1 \partial_\nu\varphi_1 + \partial_\mu v \partial_\nu v + v^2 \partial_\mu\varphi_2 \partial_\nu\varphi_2\,, \quad \gamma_\mu = k_1 \partial_\mu\varphi_1 + k_2 \partial_\mu\varphi_2, \tag{4.8}$$

where $\partial_\mu \equiv \frac{\partial}{\partial\xi^\mu}$.

JHEP07(2009)106

After some algebra, the DBI part of the action simplifies to:

$$S_{DBI}=-T_{D2}\int dtdzd\theta\left\{\frac{1}{Z_1^2}\big(\mathcal{F}_{z\theta}-\gamma_\theta(\mathcal{F}_{tz}-1)\big)^2+\frac{Z_2}{Z_1}\Delta_{\theta\theta}\Big[2(1-\mathcal{F}_{tz})-Z_3(\mathcal{F}_{tz}-1)^2\Big]\right\}^{1/2}, \tag{4.9}$$

while the WZ piece of the action takes the form

$$S_{WZ}=T_{D2}\int dtdzd\theta\left[(1-\mathcal{F}_{tz})\frac{\gamma_\theta}{Z_1}+\mathcal{F}_{z\theta}\left(\frac{1}{Z_1}-1\right)\right]. \tag{4.10}$$

For a supersymmetric configuration ($\mathcal{F}_{tz}=1$) we have

$$S_{\mathcal{F}_{tz}=1}=S_{DBI}+S_{WZ}=-T_{D2}\int dtdzd\theta\,\mathcal{F}_{z\theta} \tag{4.11}$$

The foregoing supertube carries D0 and F1 "electric" charges, given by

$$N_1^{ST}=\frac{T_{D2}}{T_{D0}}\int dzd\theta\mathcal{F}_{z\theta}, \qquad N_3^{ST}=\frac{1}{T_{F1}}\int d\theta\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}. \tag{4.12}$$

The Hamiltonian density is:

$$\mathcal{H}|_{\mathcal{F}_{tz}=1}=\left[\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\mathcal{F}_{tz}-\mathcal{L}\right]_{\mathcal{F}_{tz}=1}=T_{D2}\mathcal{F}_{z\theta}+\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}. \tag{4.13}$$

One can easily integrate this to get the total Hamiltonian of the supertube[9] (we assume constant charge density $\mathcal{F}_{z\theta}$)

$$\int dzd\theta\;\mathcal{H}|_{\mathcal{F}_{tz}=1}=N_1^{ST}+N_3^{ST}. \tag{4.14}$$

Thus the energy of the supertube is the sum of its conserved charges which shows that the supertube is indeed a BPS object.

Now choose a static *round supertube* profile $u'=v'=\varphi_2'=0$, $\varphi_1=\theta$. One then has:

$$\gamma_\theta=k_1=J\,\frac{u^2}{(u^2+v^2)^2}, \qquad \Delta_{\theta\theta}=u^2 \tag{4.15}$$

and the supertube "electric" charges are:

$$N_1^{ST}=n_2^{ST}\mathcal{F}_{z\theta}, \qquad N_3^{ST}=n_2^{ST}\frac{Z_2u^2}{\mathcal{F}_{z\theta}}. \tag{4.16}$$

So we find

$$N_1^{ST}N_3^{ST}=(n_2^{ST})^2u^2\,Z_2. \tag{4.17}$$

This is an important relation in that it fixes the location of the supertube in terms of its intrinsic charges.

[9] See appendix D for details about our units and conventions.

This computation was used in [43] to study the merger of a supertube and a black hole. In particular, a supertube can merge with a black hole if and only if $N_1^{ST} N_3^{ST} \leq (n_2^{ST})^2 N_2$, where N_2 is the number of D4 branes in the black hole. Moreover, the supertube will "crown" the black hole at "latitude", $\vartheta = \alpha$, given by:

$$\sin\alpha = \sqrt{\frac{N_1^{ST} N_3^{ST}}{(n_2^{ST})^2 N_2}}\,. \tag{4.18}$$

One can also show that one cannot violate chronology protection by throwing a supertube into the black hole, that is, one cannot over-spin the black hole and that the bound $J^2 \leq N_1 N_2 N_3$ is preserved after the merger.

4.2 Supertubes in a black-ring background

We now repeat the foregoing analysis in the background of a supersymmetric black ring where there will be new physical effects due to the interaction between the dipole charges of the black ring and the dipole charge of the supertube. We will also examine the symmetric merger of the supertube with the black ring and show that chronology protection is not violated. In section 4.4 we will perform a more general analysis by considering a probe supertube that has three charges and two dipole charges.

4.2.1 The black-ring solution

The three-charge, three-dipole charge black ring solution [20, 30, 42, 48, 49] in a IIA duality frame where the ring has D0, D4 and F1 electric charges and D6, D2 and NS5 dipole charges is given by:

$$\begin{aligned} ds^2 &= -(Z_2 Z_1)^{-1/2} Z_3^{-1} (dt+k)^2 + (Z_2 Z_1)^{1/2} ds^2_{\mathbb{R}^4} + (Z_2 Z_1)^{1/2} Z_3^{-1} dz^2 + Z_2^{-1/2} Z_1^{1/2} ds^2_{T^4}, \\ e^{2\Phi} &= Z_2^{-1/2} Z_1^{3/2} Z_3^{-1}, \\ B &= (Z_3^{-1} - 1) dt \wedge dz + Z_3^{-1} k \wedge dz - B^{(3)} \wedge dz\,, \end{aligned} \tag{4.19}$$

for the NS-NS fields, and

$$C^{(1)} = (Z_1^{-1} - 1) dt + Z_1^{-1} k - B^{(1)}, \tag{4.20}$$

$$C^{(3)} = Z_3^{-1} dt \wedge k \wedge dz - Z_3^{-1} (dt + k) \wedge B^{(1)} \wedge dz + B^{(3)} \wedge dt \wedge dz - \gamma_1 \wedge dz\,, \tag{4.21}$$

for the R-R fields. The one-forms, $B^{(I)}$, are the potentials defined in section 2.1 with $dB^{(I)} = \Theta^{(I)}$. These fields are the magnetic sources of the ring. The two-form, γ_1, must satisfy:

$$d\gamma_1 = \star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)}\,. \tag{4.22}$$

We use the canonical coordinates that are adapted to the symmetries of the black ring in the flat metric of the $\mathbb{R}^4$ base [48]:

$$ds^2_{\mathbb{R}^4} = g_{\mu\nu} dy^\mu dy^\nu = \frac{R^2}{(x-y)^2} \left(\frac{dy^2}{y^2 - 1} + (y^2 - 1) d\varphi_1^2 + \frac{dx^2}{1 - x^2} + (1 - x^2) d\varphi_2^2 \right). \tag{4.23}$$

We will also use the orientation: $\epsilon_{yx\varphi_1\varphi_2} = 1$. In these coordinates, the black ring horizon is located at $y \to -\infty$. It is useful to recall that the change of coordinates:

$$x = -\frac{u^2 + v^2 - R^2}{\sqrt{((u-R)^2 + v^2)((u+R)^2 + v^2)}}\,, \qquad y = -\frac{u^2 + v^2 + R^2}{\sqrt{((u-R)^2 + v^2)((u+R)^2 + v^2)}} \tag{4.24}$$

takes one back to the standard flat metric on $\mathbb{R}^2 \times \mathbb{R}^2$ (4.5) parameterized by (u, φ_1) and (v, φ_2) with the ring horizon at $u = R, v = 0$.

The warp factors Z_I are

$$Z_I = 1 + \frac{\overline{Q}_I}{2R^2}(x - y) - \frac{C_{IJK}}{2}\frac{q^J q^K}{4R^2}(x^2 - y^2), \tag{4.25}$$

where $\overline{Q}_I$ are what we refer to as "constituent charges" of the black ring, and differ from the charges measured at infinity. The angular momentum vector is given by

$$\begin{aligned} k &= k_1 d\varphi_1 + k_2 d\varphi_2 \\ &= -\big((y^2 - 1)\,(C(x+y) + D) - A(y+1)\big)\, d\varphi_1 - \big((x^2 - 1)\,(C(x+y) + D)\big)\, d\varphi_2 \end{aligned} \tag{4.26}$$

with $A = (q^1 + q^2 + q^3)/2$, $D = (q^1\overline{Q}_1 + q^2\overline{Q}_2 + q^3\overline{Q}_3)/8R^2$ and $C = -q^1q^2q^3/8R^2$. The vector fields, $B^{(I)}$, are given by

$$B^{(I)} = \frac{q^I}{2}\left((y + d)d\varphi_1 - (x + c)d\varphi_2\right)\,. \tag{4.27}$$

The constants c and d are locally pure gauge and are not fixed by the equations of motion. Indeed, because the ring carries a magnetic current there will Dirac strings in any attempt at a global definition of $B^{(I)}$. In the $(u, v, \varphi_1, \varphi_2)$ coordinate patch, defined by (4.24), the vector fields, $B^{(I)}$, are potentially singular at either $u = 0$, or $v = 0$. To remove these singularities we must have $(y + d) = 0$ at $u = 0$ and $(x + c) = 0$ at $v = 0$. From (4.24) we see that this unambiguously requires $d = +1$ but that one has $x = +1$ for $v = 0, u < R$ and $x = -1$ for $v = 0, u > R$ and so to remove the Dirac strings we must take:

$$d = +1\,,\;\; c = -1 \quad \text{inside the ring}\,; \qquad d = +1\,,\;\; c = +1 \quad \text{outside the ring}\,. \tag{4.28}$$

The coordinates (x, φ_2) in fact define a Gaussian two-sphere around the ring and the choices (4.28) represent the familiar gauge field patches surrounding a magnetic monopole. In the following we will set $d = 1$ and retain c with the understanding that it is to be chosen as in (4.28).

The two-form γ_1 in $C^{(3)}$ has the form $\gamma_1 = f(x, y)d\varphi_1 \wedge d\varphi_2$ where

$$f(x, y) = -\frac{\overline{Q}_2}{2}\frac{1 - xy}{x - y} + \frac{q_1q_3}{4}\left[\frac{(1 - xy)(x + y)}{x - y} + cy - dx\right] + f_0. \tag{4.29}$$

where f_0 is another integration constant. It is shown in appendix A that γ_1 satisfies (4.22).

We want to stress that our conventions are such that

$$\overline{Q}_I = \overline{N}_I \qquad \text{and} \qquad q_I = n_I \tag{4.30}$$

where $\overline{N}_I$ and n_I are integers and specify the number of "electric" and "dipole" D-branes comprising the black ring. It is also useful to note that the angular momentum of the black ring is related to its dipole charges by

$$J = 4(q_1 + q_2 + q_3)R\,. \tag{4.31}$$

4.2.2 The black ring as a solution with a Gibbons-Hawking base

Since Gibbons-Hawking (GH) geometries play an important role in bubbled solutions, and in our discussion here, it is useful to re-write the foregoing solution in terms of these geometries. The change of variables between the ordinary flat $\mathbb{R}^4$ coordinates $(u, \varphi_1, v, \varphi_2)$ and the GH coordinates (ψ, r, χ, ϕ):

$$r = \frac{1}{4}\left(u^2 + v^2\right)\,, \qquad \chi = 2\arctan\frac{u}{v}\,, \qquad \psi = 2\,\varphi_1\,, \qquad \phi = -\left(\varphi_2 + \varphi_1\right)\,, \tag{4.32}$$

and recall that u and v are related to x and y by (4.24). The metric in the new coordinates is:

$$ds^2_{\mathbb{R}^4} = r\left(d\psi + (\cos\chi + 1)d\phi\right)^2 + \frac{1}{r}\left(dr^2 + r^2 d\chi^2 + r^2\sin^2\chi d\phi^2\right) \tag{4.33}$$

The black ring solution is written in terms of eight harmonic functions V, L_I, K^I and M [14, 30–33]. However, as we noted in the last subsection, the black ring has a monopolar magnetic field and so we need two patches that are related by a gauge transformation. Remembering that the vector potentials in solutions with a GH base are given by

$$B^{(I)} = V^{-1}K^I(d\psi + A) + \xi^I\,, \tag{4.34}$$

one can easily identify the K^I that give these fields, and observe that changing the patch from $c{=}-1$ to $c{=}+1$ corresponds, in the GH solution, to the gauge transformation:

$$\begin{aligned} &K^I \to K^I + c^I V\,, \qquad\qquad L_I \to L_I - C_{IJK}c^J K^K - \frac{1}{2}C_{IJK}c^J c^K V, \\ &M \to M - \frac{1}{2}\,c^I\,L_I + \frac{1}{12}\,C_{IJK}\left(V\,c^I\,c^J\,c^K + 3\,c^I\,c^J\,K^K\right)\,, \end{aligned} \tag{4.35}$$

with $c^I = q^I/2$. Thus, we can now completely specify the eight harmonic functions, once we choose a patch. For $c = -1$, we have

$$\begin{aligned} V &= \frac{1}{r}\,, & K^I &= -\frac{q_I}{2|\vec{r} - \vec{r}_{BR}|}\,, \\ L_I &= 1 + \frac{\overline{Q}_I}{4|\vec{r} - \vec{r}_{BR}|}\,, & M &= -\frac{J}{16|\vec{r} - \vec{r}_{BR}|} + \frac{J}{16R}\,, \end{aligned} \tag{4.36}$$

and for $c = +1$ they become

$$\begin{aligned} V &= \frac{1}{r}\,, & K^I &= -\frac{q_I}{2|\vec{r} - \vec{r}_{BR}|} + \frac{q_I}{2r}\,, \\ L_I &= 1 + \frac{\overline{Q}_I + C_{IJK}q^J q^K}{4|\vec{r} - \vec{r}_{BR}|} - \frac{C_{IJK}q^J q^K}{8r}\,, & M &= -\frac{J + q^I\overline{Q}_I + 3q^1q^2q^3}{16|\vec{r} - \vec{r}_{BR}|} - \frac{q^1q^2q^3}{16r}\,. \end{aligned}$$

As noted earlier, these formulae *define* the GH charges of the black ring and these, in turn, define the electric charges of the four-dimensional black hole corresponding to the ring. The electric GH charges Q_I^{GH} are four times the coefficients of the pole at the location of the ring in the L_I functions, the GH dipole charges q_I^{GH} are minus two times the coefficients of the pole in the K^I functions, and the GH angular momentum J^{GH} is minus sixteen times the coefficient of the pole in M (we use the conventions of [14]). Thus, we have:

$$Q_I^{GH} = \overline{Q}_I\,, \qquad q_I^{GH} = q_I\,, \qquad J^{GH} = J \tag{4.37}$$

for $c = -1$ and

$$Q_I^{GH} = \overline{Q}_I + C_{IJK}q^J q^K\,, \qquad q_I^{GH} = q_I \qquad J^{GH} = J + q^I\overline{Q}_I + 3q^1q^2q^3 \tag{4.38}$$

for $c = +1$.

The dipole charges are patch-independent, but the GH electric charges and the GH angular momentum are *gauge dependent* notions, and are different in different patches. This will be important in the following discussion.

4.2.3 Probing the black ring with two-charge supertubes

We now probe the black ring background with a two-charge supertube [1, 50]. The calculation proceeds in much the same way as for the supertube in a black hole background. As before, we parameterize the tube by (t, z, θ), and define an *a priori* arbitrary supertube profile in $\mathbb{R}^4$ by $\vec{y}(\theta)$. Since we are ultimately going to consider a supertube that winds multiple times around the ring direction it will be convenient to take $\theta \in (0, 2\pi n_2^{ST})$ where n_2^{ST} will become this winding number. Thus the supertube will have a dipole charge proportional to n_2^{ST}, and two net charges proportional to N_1^{ST} and N_3^{ST}. Its action is a sum of a DBI and a WZ term

$$\begin{aligned} S = S_{DBI} + S_{WZ} &= -T_{D2}\int dtdzd\theta e^{-\Phi}\sqrt{-\det(\widetilde{G}_{ab} + \widetilde{B}_{ab} + \mathcal{F}_{ab})} \\ &\quad + T_{D2}\int dtdzd\theta\left(\widetilde{C}^{(3)}_{tz\theta} + \widetilde{C}^{(1)}_t(\widetilde{B}_{z\theta} + \mathcal{F}_{z\theta}) + \widetilde{C}^{(1)}_\theta(\widetilde{B}_{tz} + \mathcal{F}_{tz})\right) \end{aligned} \tag{4.39}$$

For the supersymmetric configuration one once again finds that $\mathcal{F}_{tz} = 1$ and if one imposes this *ab initio* then one again obtains (4.11), (4.12) and (4.13) and hence the BPS relation for the supertube. The expression for the derivative of the action with respect to $\mathcal{F}_{tz}$ evaluated at $\mathcal{F}_{tz} = 1$ can be most convenient expressed as:

$$\left(\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1} + T_{D2}\left(B^{(1)}_{\varphi_1}\varphi_1' + B^{(1)}_{\varphi_2}\varphi_2'\right)\right)\left(\mathcal{F}_{z\theta} + \left(B^{(3)}_{\varphi_1}\varphi_1' + B^{(3)}_{\varphi_2}\varphi_2'\right)\right) = T_{D2}Z_2 g_{\mu\nu}y'^{\mu}y'^{\nu}\,, \tag{4.40}$$

where $'$ denotes the derivative with respect to θ. As for the black hole [43, 44], one can reinterpret this in terms of charge densities and arrive at a generalization of the constraint (4.17) that relates the charges to the radius of the supertube. Note that the condition (4.40) is local and to get a relation similar to (4.17) on has to integrate over the profile of the supertube. There is an important new feature here in that there is a contribution from

JHEP07(2009)106

the interactions of the dipole charges of the supertube and background. This appears through the pull-back of the $B^{(I)}$ to the world-volume of the supertube and it gives an added contribution to the basic supertube charges to yield what we will refer to as the local *effective charges* of the supertube. We will show in section 4.4 that this also happens when supertubes are placed in three-charge solutions with a GH base.

It is also important to remember that the Wess-Zumino action of the supertube is only invariant under local small gauge transformations, but is not necessarily invariant under *large gauge transformations.* Indeed, the black ring is a magnetic object, and as such the gauge fields, $B^{(I)}$ are not defined globally but on patches. Their values, and the value of the supertube action, differ from patch to patch by what can be thought of as the effect of a large gauge transformation.

More explicitly, the action depends on the Wilson lines of these gauge fields taken around latitudes of the two-sphere that surrounds the black ring (which is the equivalent of the sphere that contains a monopole charge). The value of these Wilson loops may then be defined using Stokes theorem as the integral of the magnetic flux coming from the black ring through the section of the sphere surrounded by the Wilson line. There is, however, an obvious ambiguity: does one integrate the flux over the upper or the lower cap of the sphere? The difference is, of course, the monopole charge inside the sphere multiplied by the number of times the Wilson loop winds around the latitude circle. These ambiguities will manifest themselves in the definitions of the constituent charges of the supertube.

To analyze the physics of the merger, we consider a supertube embedded in spacetime along the curve $\vec{y}(\theta)$ given by:

$$\varphi_1 = -\theta \,, \qquad \varphi_2 = -\nu\,\theta \tag{4.41}$$

x and y being at fixed values. The projections of the supertube in the (y, φ_1) and (x, φ_2) planes are circular, with winding numbers n_2^{ST} and νn_2^{ST} respectively. For $\nu = 0$, the supertube is circular and simply winds around the plane of the ring n_2^{ST} times. For $\nu \neq 0$, the details of the winding depend upon the equilibrium position of the supertube. We also assume, for simplicity, that the charge densities of the tube are independent of θ. Under these assumptions the condition (4.40) becomes:

$$\left[N_1^{ST} - \frac{1}{2} n_2^{ST} n_3 \left(y + 1 - \nu(x+c)\right)\right] \left[N_3^{ST} - \frac{1}{2} n_2^{ST} n_1 \left(y + 1 - \nu(x+c)\right)\right] = \left(n_2^{ST}\right)^2 Z_2 \frac{R^2}{(x-y)^2} \left((y^2-1) + \nu^2(1-x^2)\right) . \tag{4.42}$$

We will call this equation the radius relation. Note that this equation is invariant under the exchange of N_1, n_1 with N_3, n_3, as expected by U-duality. Comparing this constraint to the one for a black hole background (4.17), we see that the charges of the supertube are enhanced to their *effective charges* via the interactions of the dipole charges. This is an important result that we will discuss further in the subsequent sections.

To get a better idea of the supertube configuration in the black-ring geometry it is instructive to examine the supertube as it approaches the horizon ($y \to -\infty$). In this limit, the physical metric along the horizon becomes:

$$ds_3^2 = \left(C^2R^4\right)^{1/3}\left[\left(64\,C^2R^4\right)^{-1}\mathcal{M}\,d\varphi_1^2 + \left(d\alpha^2 + \sin^2\alpha\,(d\varphi_1 + d\varphi_2)^2\right)\right], \tag{4.43}$$

where we have set $x = \cos\alpha$, and the parameter, $\mathcal{M}$, is proportional to the square of the black-ring entropy

$$S = \pi\sqrt{\mathcal{M}}, \tag{4.44}$$

and is given by

$$\mathcal{M} = 2n_1n_2\overline{N}_1\overline{N}_2 + 2n_1n_3\overline{N}_1\overline{N}_3 + 2n_2n_3\overline{N}_2\overline{N}_3 - (n_1\overline{N}_1)^2 - (n_2\overline{N}_2)^2 - (n_3\overline{N}_3)^2 - 4n_1n_2n_3J\,, \tag{4.45}$$

where J is the "intrinsic" angular momentum of the ring, and is given by the difference between the two angular momenta of the five-dimensional solution:

$$J = J_1 - J_2 = 4(n_1 + n_2 + n_3)\,R\,. \tag{4.46}$$

The topology of the horizon is $S^2 \times S^1$, but observe that for a supertube that winds according to (4.41), the winding around the horizon is determined by

$$\varphi_1 = -\theta\,, \qquad \varphi_1 + \varphi_2 = -(\nu+1)\,\theta\,. \tag{4.47}$$

The supertube thus enters the horizon by winding around the S^1 but enters at a point on the S^2 if and only if $\nu = -1$. Otherwise it winds around the S^1 and "crowns" the S^2 by winding $(\nu+1)$ times around a latitude determined by x.

If we now examine the constraint (4.42) and send $y \to -\infty$ the supertube will merge with the black ring and the constraint (4.42) will become the merger condition:

$$N_1^{ST}n_1 + N_3^{ST}n_3 - \overline{N}_2 n_2^{ST} = n_2^{ST}n_1n_3((1+c) - (\nu+1)(x+c))\,. \tag{4.48}$$

More explicitly, this condition be written as:

$$N_1^{ST}n_1 + N_3^{ST}n_3 - \overline{N}_2 n_2^{ST} = n_2^{ST}n_1n_3\,(\nu+1)(1-x) \qquad \text{for} \quad c = -1\,. \tag{4.49}$$

$$N_1^{ST}n_1 + N_3^{ST}n_3 - \overline{N}_2 n_2^{ST} = n_2^{ST}n_1n_3\,(2 - (\nu+1)(1+x)) \quad \text{for} \quad c = +1\,. \tag{4.50}$$

The relation (4.48) is simply the analogue of the equation giving the merging angle for the supertube in a black-hole background (4.18). In particular, as depicted in figure 1, it determines the value of x (which corresponds to an angular variable on the horizon) at which a supertube with a given set of charges enters the black ring horizon. Since $-1 \le x \le +1$, this restricts the permissible charges of supertubes that merge with a given black ring.

We can see that the radius relation (4.42) and the merger condition (4.48) depend both on the gauge choice (by an x-independent factor) and also on $\nu + 1$. We can understand this gauge dependance in a physical way: the gauge choice corresponds to a choice for the location of the Dirac string. In other words, the gauge dependance comes from the fact that the tube feels the presence of the Dirac string of the background. Increasing x then

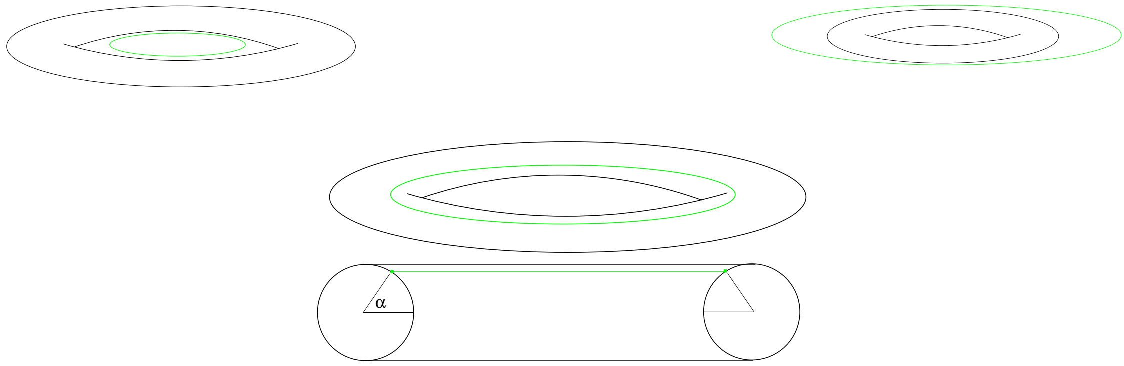

Figure 1. Different black ring and supertube configurations for different values of the supertube charges. In the first picture, the charges of the tube are too small, and hence the tube it is too small, and passes inside the ring. In the second one, the tube is too large and passes on the outside of the ring. In the third picture, the size of the tube is in the correct range for the merger to be possible. The angle α of the merger depends on the tube charges according to (4.48).

corresponds to the supertube wrapping, for $c = -1$, or not wrapping, for $c = +1$ the Dirac string, as can be seen in figure 2.

More precisely, if we choose $c = -1$, that is if we choose the Dirac string to extend from the ring location to infinity, then we can put the tube everywhere except on the Dirac string. If we put it at $x = 1$, the ϕ circle becomes degenerate and indeed in (4.49) the ν dependance disappears. This is expected, because $\nu + 1$ is the winding number of the tube around a contractible circle. When the size of this circle is zero, the winding should be irrelevant, which is indeed what happens.

If we now change the location of the ring to approach $x = -1$ without changing the gauge, the tube winds $\nu + 1$ times around the Dirac string; this winding is physically-relevant, and hence, as expected, equation (4.49) depends on ν when $x \to 1$. However, if we change the gauge to move the Dirac string to the inside of the ring, we can see that when the tube is at $x = 1$, where the ϕ circle is degenerate, the winding number is again irrelevant; as expected the merger formula is again independent of ν. We should also note that for the particular value $\nu = -1$, the supertube never wraps the Dirac string, and hence the merger condition does not depend upon x.

In section 5 we will examine the details of such a merger and discuss chronology protection and black hole thermodynamics during mergers.

4.3 The black ring background: comparing the DBI analysis with supergravity

We now turn to the main purpose in this section: the relation between the merger conditions obtained from supergravity and from the DBI analysis, and the relation between the GH and the DBI charges of the supertube.

Let begin with the supergravity side. The supergravity solution corresponding to one black ring and one supertube is given as usual the eight harmonic functions V, L_I, K^I and

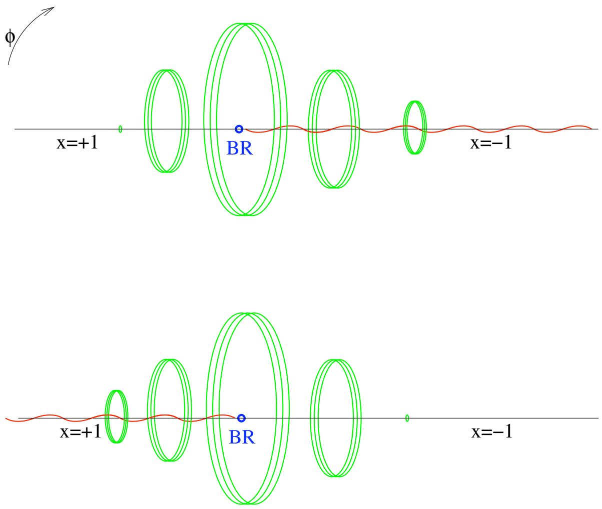

Figure 2. The black ring (in blue) with supertubes (in green) at various positions in the $\mathbb{R}^3$ base of the Gibbons-Hawking space. The black ring is point-like but the tube is point-like only if it lies on the axis $x = \pm 1$. Otherwise, it winds $\nu + 1$ times the ϕ circle. On the left, the Dirac string starts from the ring and extends to infinity. On the right, the Dirac string extends between the center of the space and the ring location.

M. The poles of this functions at the location of the ring and of the tube are

$$K_1 = -\frac{q_1}{2|\vec{r} - \vec{r}_{BR}|}, \quad 2M = -\frac{J^{GH}}{8|\vec{r} - \vec{r}_{BR}|} - \frac{J^{GH,ST}}{8|\vec{r} - \vec{r}_{ST}|} \tag{4.51}$$

where Q^{GH} are the GH charges of the black ring defined in section 4.2.2, and $Q^{GH,ST}$ are the GH charges of the supertube defined in the same way. Recall once again that the GH charges depend upon the choice of patch, as in (4.37) and (4.38), and the GH charges of both the ring and the tube transform consistently between the patches.

To obtain the merger condition from supergravity observe that the bubble (or integrability) equations (2.14) contain a term in which the $E_{7(7)}$ symplectic product of the supertube and black ring GH charge vectors is divided by their separation. Hence, these objects only merge if this symplectic product is zero.[10] Explicitly, this gives[11]

$$N_1^{GH,ST} n_1 + N_3^{GH,ST} n_3 - N_2^{GH} n_2^{ST} = 0. \tag{4.52}$$

Note that the GH charges of the ring and of the tube are gauge dependent, but the symplectic product is invariant.

[10] One could also imagine in principle the existence of a scaling solution, where the distances in $\mathbb{R}^3$ between the ring, supertube and the center of Taub-NUT go together to zero. In such a solution the ring and the supertube would be spinning very rapidly in opposite directions, which is likely to introduce closed timelike curves. We leave its exploration for future work.

[11] As noted in (4.30), we have adopted a set of conventions in which the supergravity charges, Q^{ST}, are the same as the integer charges.

To compare the GH merger conditions (4.52) to the merger conditions obtained in the previous section using the DBI action, one should recall that this condition describes only those supertubes that correspond to point sources on the $\mathbb{R}^3$ of the GH base. That is, the supertubes are embedded into $\mathbb{R}^4$ so as to wind around the GH fiber, and thus preserve the same triholomorphic U(1) isometry as the black ring. From (4.41) and (4.32) we see that the winding numbers of the supertube in the GH patch are given by $(1, \nu+1)$. (Remember that ψ has period 4π.) Thus a supertube is point-like in the $\mathbb{R}^3$ if and only if it has either $\nu = -1$ or it lies on the polar axis with $x = \pm 1$. We therefore restrict ourselves to mergers with $x = \pm 1$ for any value of ν, or mergers with $\nu = -1$.

For $x = 1$, we need to be on the patch $c = -1$, and (4.48) gives:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = 0 \; . \tag{4.53}$$

For $x = -1$, we need to be on the patch $c = +1$, and thus have:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = 2 n_2^{ST} n_1 n_3 \; . \tag{4.54}$$

But using the relation (4.38), we can rewrite it as

$$N_1^{ST} n_1 + N_3^{ST} n_3 - N_2^{GH} n_2^{ST} = 0 \tag{4.55}$$

on both patches. The extra term in (4.54) is simply the shift in N^{GH} induced by changing patches. Thus, if we identify the DBI charge of the supertube with the GH charge of the corresponding supergravity solution,

$$N_I^{ST} = N_I^{GH,ST}, \tag{4.56}$$

we have a perfect agreement between the supergravity approach (4.52) and the DBI approach (4.55).

The supertubes with $\nu = -1$ do not wrap the ϕ circle of the $\mathbb{R}^3$ base of the GH space, and thus are point-like in this base for any value of x, and they source a supergravity solution with a GH base for any location. Moreover, since these tubes do not wrap the Dirac string, the merger relations become x independent. Equations (4.49) and (4.50) then become

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = 0 \qquad \text{for} \quad c = -1 \, , \tag{4.57}$$

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = 2 n_2^{ST} n_1 n_3 \qquad \text{for} \quad c = +1 \, , \tag{4.58}$$

which once again can be re-written as

$$N_1^{ST} n_1 + N_3^{ST} n_3 - N_2^{GH} n_2^{ST} = 0 \, . \tag{4.59}$$

Hence we arrive at the same conclusion as for supertubes at $x = \pm 1$: the DBI charges of the supertube give the GH charges of the corresponding supergravity solution:

$$N_I^{ST} \equiv N_I^{GH,ST} \, . \tag{4.60}$$

4.4 Black rings and three-charge two-dipole-charge supertubes

One can generalize the foregoing discussion of mergers to examine a three-charge, two dipole charge supertube [43] merging with a generic black ring. This can be done both in the probe approximation, using the DBI action, and in the exact supergravity solution. This supertube is more general than the two-charge supertube, and although it does not source a smooth supergravity solution in any duality frame, it can be used to study rather more general classes of mergers.

The best duality frame to study this merger is that in which the three-charge supertube is a dipolar D6-brane carrying electric D4, D0 and F1 charges. We take our tube to be along the $(x_1, x_2, x_3, x_4, z, \vec{y}(\theta))$, where $\vec{y}(\theta)$ describes a closed curve in the non-compact space. As before, we take $\theta \in (0{,}2\pi n_1^{ST})$ with n_1^{ST} being the winding number of the supertube which is also its D6 dipole charge. We introduce world-volume electric fields: $\mathcal{F}_{z\theta}$, $\mathcal{F}_{tz}$, $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$. where $\mathcal{F}_{tz}$ and $\mathcal{F}_{z\theta}$ generate the F1 and D4 charges respectively and $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$ are needed for the D0 charge. The integer charges are given by

$$N_1^{ST} = N_{D0} = \frac{1}{2\pi}\int d\theta\, \mathcal{F}_{z\theta}\mathcal{F}_{12}\mathcal{F}_{34}\,, \tag{4.61}$$

$$N_2^{ST} = N_{D4} = \frac{1}{2\pi}\int d\theta\, \mathcal{F}_{z\theta}\,, \tag{4.62}$$

$$N_3^{ST} = N_{F1} = \frac{1}{2\pi}\int d\theta\, \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}\,, \tag{4.63}$$

$$n_2^{ST} = n_{D2} = n_1^{ST}\mathcal{F}_{12}\mathcal{F}_{34}\,. \tag{4.64}$$

Note that we can take the D4 dipole moments and D2 charges of the tube to be zero by taking $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$ to be traceless. Supersymmetry requires that $\mathcal{F}_{tz} = 1$ and $\mathcal{F}_{12} = \mathcal{F}_{34}$ [43], and then one can show that

$$\mathcal{H}|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}} = T_{D6}\mathcal{F}_{z\theta}\mathcal{F}_{12}\mathcal{F}_{34} + T_{D6}\mathcal{F}_{z\theta} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}}\,, \tag{4.65}$$

or equivalently

$$\int d^4x dz d\theta\, \mathcal{H}|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}} = N_1^{ST} + N_2^{ST} + N_3^{ST}\,, \tag{4.66}$$

where $\mathcal{H}$ is the energy per unit five-dimensional volume.

As before, we will assume constant charge densities on the supertube worldvolume and the interesting physical condition that generalizes (4.42) comes from the variation that define the F1-charge, N_3^{ST}:

$$\begin{aligned}\left[N_3^{ST} - \frac{1}{2}(n_1^{ST}n_2 + n_2^{ST}n_1)(y+1-\nu(x+c))\right]\left[N_2^{ST} - \frac{1}{2}n_1^{ST}n_3(y+1-\nu(x+c))\right] = \\ n_1^{ST}\left(n_1^{ST}Z_1 + n_2^{ST}Z_2\right)\frac{R^2}{(x-y)^2}((y^2-1)+\nu^2(1-x^2))\,.\end{aligned} \tag{4.67}$$

Note that, using $n_1^{ST}N_1^{ST} = n_2^{ST}N_2^{ST}$, there is a symmetry between (D0,D6) and (D4,D2) charges and dipole moments, as expected from U-duality. However since the tube has no NS5 dipole moment, there is no exchange symmetry between the F1 charge and other charges.

One can extract the merger condition from this as before and one finds that, for a merger with a black ring, (4.48) generalizes to:

$$n_1 N_1^{ST} + n_2 N_2^{ST} + n_3 N_3^{ST} - n_1^{ST}\overline{N}_1 - n_2^{ST}\overline{N}_2 = n_3\left(n_1 n_2^{ST} + n_2 n_1^{ST}\right)\left((1+c) - (\nu+1)(x+c)\right). \tag{4.68}$$

When the three-charge supertube respects the GH isometry ($x = \pm 1$ for any ν or $\nu = -1$ for any x), one can also describe this merger in supergravity. The solution is given by the same harmonic functions as in (4.51), except that now K_1 and L_2 also have poles at the supertube location:

$$K_1 \to -\frac{q_1^{GH}}{2|\vec{r} - \vec{r}_{BR}|} - \frac{q_1^{GH,ST}}{2|\vec{r} - \vec{r}_{ST}|}\,, \quad L_2 \to \frac{Q_2^{GH}}{4|\vec{r} - \vec{r}_{BR}|} + \frac{Q_2^{GH,ST}}{4|\vec{r} - \vec{r}_{ST}|}\,. \tag{4.69}$$

One can see that equation (4.68) is equivalent to the vanishing of the $E_{7(7)}$ symplectic product of the GH charges of the black ring and those of the three-charge supertube, and hence the merger conditions obtained from supergravity and from the Born-Infeld analysis of the three-charge supertube are the same. The subtleties associated to the dependence of the charges upon the patch are identical to those for the two-charge supertube, and we will not discuss them again.

4.5 Supertubes in a general solution with a Gibbons-Hawking base

We now consider two-charge supertubes probing a general three-charge BPS solution with a Gibbons-Hawking base and we will again work in the D0-D4-F1 duality frame. The general BPS solution with three charges and three dipole charges and a GH base is given in sections 2.1 and 2.2 and we proceed as we did for the black-hole and black-ring backgrounds in sections 4.1 and 4.2. We denote the supertube coordinates as ξ^0, ξ^1 and $\xi^2 \equiv \theta$ and consider the simplified case of a circular supertube along the U(1) fiber of the GH base:

$$\xi^0 = t\,, \qquad \xi^1 = z\,, \qquad \theta = \psi\ . \tag{4.70}$$

The supertube action (4.40) takes the explicit form

$$\begin{aligned} S = T_{D2}\int d^3\xi \Bigg\{ &\left[\left(\frac{1}{Z_1} - 1\right)\mathcal{F}_{z\theta} + \frac{K^3}{Z_1 V} + \left(\frac{\mu}{Z_1} - \frac{K^1}{V}\right)(\mathcal{F}_{tz} - 1)\right] \\ &- \left[\frac{1}{V^2 Z_1^2}\left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + V Z_1 Z_2 (1-\mathcal{F}_{tz})(2 - Z_3(1-\mathcal{F}_{tz}))\right]\right]^{1/2}\Bigg\}. \end{aligned} \tag{4.71}$$

For $\mathcal{F}_{tz} = 1$ the tube is supersymmetric and, as before, the Hamiltonian density is the sum of the charge densities (4.13). Due to the supersymmetry there is a constraint similar to (4.42), which determines the location of the supertube in terms of its charges

$$\left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + \frac{K^1}{V}\right] = (n_2^{ST})^2\frac{Z_2}{V}\,, \tag{4.72}$$

where the charges are still defined by (4.12).

4.6 Gibbons-Hawking backgrounds: comparing the DBI analysis with supergravity

Equation (4.72) determines the position of a supertube in an arbitrary three-charge background with a triholomorphic U(1) isometry. Since both the supertube and the background preserve this isometry, their fully back-reacted supergravity solution will have a Gibbons-Hawking base, and its form is well-known. Hence, one can compare (4.72) to the corresponding condition coming from the supergravity analysis of the supertube, and confirm that supertubes that are solutions of the Born-Infeld action always give rise to smooth supergravity solutions.

To do this, it is useful to remember that in any Gibbons-Hawking solution the singularities in the harmonic functions K_2, L_1, L_3 and M at the supertube location are given by (4.51). If one now takes equation (3.9) for a supertube with charges $Q_1^{GH,ST}, Q_3^{GH,ST}$ and q_2^{ST} and uses the asymptotic behavior of these harmonic functions near the supertube one obtains:

$$\left[Q_1^{GH,ST} - 2q_2^{ST}\frac{K^3}{V}\right]\left[Q_3^{GH,ST} - 2q_2^{ST}\frac{K^1}{V}\right] = \left(q_2^{ST}\right)^2\frac{Z_2}{V}\,. \tag{4.73}$$

Since the supergravity GH charges, $Q_1^{GH,ST}, Q_3^{GH,ST}, q_2^{ST}$, are the same as the integer charges $N_1^{GH,ST}, N_3^{GH,ST}, n_2^{ST}$, one sees that this agrees exactly with the DBI calculation.

It is interesting to observe that the DBI action only gives one equation of motion for the supertube, (4.72), while the supergravity analysis of the supertube gives two independent equations, that can be chosen to be any two of (3.4), (3.5) and (3.9). This is because in the Born-Infeld analysis the inputs are the supertube charges and dipole charge, which one first uses to find the embedding, and then one derives the angular momentum of the supertube, J^{ST}, from that solution.

By contrast, in the supergravity analysis the angular momentum of the supertube along the Gibbons-Hawking fiber appears as the coefficient of the singular part in the harmonic function M, and is one of the *inputs* of the calculation. Indeed, in supergravity one can build "supertube" solutions for any value of J_T. However most of these solutions will be singular: if J_T is too large the solutions will have closed timelike curves, and if J_T is too small the solutions will have a naked singularity.[12] Only one specific value of J_T gives a supergravity solution that is smooth and horizonless in the duality frame in which the supertube charges correspond to D1 and D5 branes.

To find this value it is most convenient to use equation (3.4), and the expansion of the harmonic functions (4.51) near the supertube location to find the supertube angular momentum as a function of the supertube charges $Q_1^{GH,ST}, Q_3^{GH,ST}$ and dipole charge q_2^{ST}:

$$J^{GH,ST} = \frac{N_1^{GH,ST} N_3^{GH,ST}}{n_2^{ST}} \tag{4.74}$$

To obtain this equation from the DBI analysis one needs to calculate the angular momentum of the supertube along the Gibbons-Hawking fiber. This calculation is partially

[12] Such a singularity might be cloaked by a Planck-sized horizon [51].

shown in appendix C[13] and gives

$$J^{ST} = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}}. \tag{4.75}$$

This indicates that when supertubes are embedded in a solution with a Gibbons-Hawking base, respecting the triholomorphic U(1) isometry of this solution, their Born-Infeld analysis gives the equations needed for the fully back-reacted supergravity solution of these supertubes to be smooth and free of closed timelike curves.

4.7 A comment on black rings in Taub-NUT and their four-dimensional charges

An interesting by-product of our results in section 4.2.2 is that a given five-dimensional supersymmetric black ring can be embedded in Taub-NUT [14, 32, 33] in many ways depending upon the choice of the gauge field for the in the magnetic flux.[14] We considered patches and gauge choices that preserve the U(1) of the GH base and this still left a free parameter, c, in (4.27). The two natural patches, with $c = +1$ and $c = -1$ have a single Dirac string, and together they provide a complete cover of the solution. Other choices of c split the Dirac strings into two parts, one at each pole of the S^2. If one compactifies the black ring down to a four-dimensional black hole then we saw that the electric charges of the black hole are given by the GH electric charges at the ring location. We also saw that the GH charges depended upon the choice of patch and if one uses the $c = +1$ or $c = -1$ then the black-hole charges are not the same as the electric charges, measured at infinity, of the five-dimensional black ring.

Hence, from a four-dimensional perspective the black ring can correspond to an infinite family of black holes, whose D2 and D0 charges are related via the gauge transformation (4.35). The effect of this transformation is to introduce Wilson lines for the gauge fields along the Taub-NUT circle at infinity, and to create or remove Dirac strings at the north or south pole of the black hole. Nevertheless, even if the four-dimensional charges depend upon the choice of gauge, the warp factors Z_I and the symplectic products that determine the metric, the field strengths, and the location of the black ring, are invariant under (4.35).

One can also take a peculiar gauge with $c = 0$ for which the solution has two Dirac strings but for this choice the four-dimensional electric charges are the same as the asymptotic charges in the five-dimensional solution [17]. On the other hand, in this gauge the D0 charge is given neither by the five-dimensional "ring angular momentum" (which was the difference between the two angular momenta in five dimensions), nor by the total angular momentum in the plane of the ring, J_1 (as assumed in [53]), but rather it is given by a combination of the five-dimensional charges and angular momenta that has no obvious interpretation in five dimensions:

$$J_{c=0} = J_1 - J_2 + \frac{1}{2} q^I \overline{Q}_I + \frac{3}{4} q^1 q^2 q^3 \,. \tag{4.76}$$

[13]For supertubes in $\mathbb{R}^4$ in the presence of arbitrary charges and dipole charges

[14]For the embedding of nonsupersymmetric black rings in Taub-NUT see [52]

It is not hard to see that all the shifts of charges brought about by gauge transformations leave the $E_{7(7)}$ quartic invariant unchanged. The entropy of the ring is still determined by this invariant [15], but now as a function of the shifted electric charges, and the shifted angular momentum. Therefore, the entropy of all the four-dimensional black holes related to the ring can be understood microscopically by an MSW analysis [54] that is done without the shift of L_0. Hence, the observation of [17] that the five-dimensional asymptotic electric charges of the black ring can be related to those of a four-dimensional black hole does not solve the discrepancy between the two microscopic descriptions of black rings[15] [15, 53].

Our analysis thus establishes that the four-dimensional charges that one uses in the $E_{7(7)}$ quartic invariant to obtain the black ring entropy, depend on the choice of patch, and one can switch between various charges (like the asymptotic charges of the ring and the intrinsic charges) by gauge transformations. Nevertheless, this transformation generically also changes the angular momentum parameter (or the D0 charge). Therefore, in trying to find the microscopic description of extremal non-BPS black rings (as was done recently in [55]) one should not focus on the fact that a certain charge appears in the quartic invariant, but rather on a gauge-independent concept like why, for a given choice of charges, does a certain angular momentum parameter appear in the quartic invariant.

5 Chronology protection

Having obtained the condition under which a supertube and a black ring can merge, both using the Born-Infeld description of supertubes, and (where appropriate) also using the supergravity solution corresponding to the merger, we now turn to verifying that supertube mergers preserve the physical properties of the black ring. For simplicity, and because it is sufficient for capturing all the relevant physics of the merger, we will primarily focus on circular embeddings for the tube (4.41).

5.1 Mergers of black rings with two-charge supertubes

We begin by considering the merger of a black ring with a two-charge supertube of arbitrary shape. To do this one must first establish what shape can the supertube have when it crosses the black ring horizon. Based on our intuition from supertubes merging with black holes [43] we expect that the supertube will be parallel to the horizon, and that it should not be possible to have a part of the supertube inside the black ring horizon and a part of it is outside.

To see this we can analyze equation (4.40) and change variables to $w = \frac{1}{y}$; the merger then happens at $w \to 0$. After some algebra one can see that for $w \to 0$ the leading divergent term in (4.40) imposes the constraint $\frac{\partial w}{\partial \theta} = 0$, which implies that the supertube is always tangent to the horizon when it merges to a black ring.

It is particularly important to examine the thermodynamics of mergers and see whether by "throwing in" supertubes one could decrease the entropy of a black ring, or overspin

[15] One might get this impression from [17].

it and introduce closed timelike curves (violating chronology protection). To do this one must determine what are the charges that a supertube brings into a ring. As we saw in the section 4.3, there are some subtleties in this determination and we cannot always add the DBI charges of the supertube to the constituent charges, the $\overline{N}$'s, of the ring. We have learned that the DBI charges have to be identified with the GH charges of the supertube, which are patch-dependent, and are not the same as the constituent ones. We have seen this explicitly from the supergravity solution for concentric mergers (when $x = \pm 1$) or alternatively when we take $\nu = -1$ so that the supertube does not wind around latitude circles and crosses the ring horizon at a point on the S^2 of the horizon. We will first focus on mergers where the supertube merges at a point on the S^2, and discuss the other ones at the end of this subsection.

The entropy of the black ring is given by $S = \pi\sqrt{\mathcal{M}}$ where $\mathcal{M}$ is defined in (4.45)

$$\mathcal{M} = 2n_1n_2\overline{N}_1\overline{N}_2+2n_1n_3\overline{N}_1\overline{N}_3+2n_2n_3\overline{N}_2\overline{N}_3-(n_1\overline{N}_1)^2-(n_2\overline{N}_2)^2-(n_3\overline{N}_3)^2-4n_1n_2n_3J\,. \tag{5.1}$$

Note that $\mathcal{M}$ is in fact the $E_{7(7)}$ quartic invariant and is therefore invariant under a gauge transformation (4.35). In terms of GH charges of the ring, we have

$$\begin{aligned}\mathcal{M} \;=\; & 2n_1n_2N_1^{GH}N_2^{GH} + 2n_1n_3N_1^{GH}N_3^{GH} + 2n_2n_3N_2^{GH}N_3^{GH} \\ & -(n_1N_1^{GH})^2 - (n_2N_2^{GH})^2 - (n_3N_3^{GH})^2 - 4n_1n_2n_3J^{GH}\,.\end{aligned} \tag{5.2}$$

From the analysis in the previous section, we know that the supertube DBI charges correspond to GH charges, and thus should be directly added to the GH charges of the ring.

To keep the expressions simple we will take the three electric and the three dipole charges of the black ring charges to be equal, we will also assume that the two electric charges of the supertube are equal, namely:

$$N_1^{GH} = N_2^{GH} = N_3^{GH} \equiv N\,, \qquad n_1 = n_2 = n_3 \equiv n\,, \qquad N_1^{ST} = N_3^{ST} \equiv \Delta N\,. \tag{5.3}$$

Then we have

$$\mathcal{M} = n^2(3N^2 - 4nJ) \tag{5.4}$$

and the charges of physical black rings satisfy: $3N^2 \geq 4nJ$.

Let Δn denote the dipole charge of the tube and ΔJ its angular momentum. The new horizon area parameter, $\widetilde{\mathcal{M}}$, after the merger is then

$$\begin{aligned}\widetilde{\mathcal{M}} \;=\; & 4\,n\,N\,(n+\Delta n)(N+\Delta N) + 2\,n^2(N+\Delta N)^2 - (n+\Delta n)^2N^2 \\ & -2\,n^2(N+\Delta N)^2 - 4\,n^2(n+\Delta n)(J+\Delta J) \\ =\; & \mathcal{M} + n\,\Delta n(3N^2 - 4nJ) \\ & -\frac{(n+\Delta n)}{\Delta n}\left[(2\,n\,\Delta N - N\Delta n)^2 + 4n^2\Delta n\left(\Delta J - \frac{(\Delta N)^2}{\Delta n}\right)\right]\,.\end{aligned} \tag{5.5}$$

We now need to remember that the angular momentum of the tube is given by (C.34)

$$\Delta J = \frac{(\Delta N)^2}{\Delta n}\,, \tag{5.6}$$

and also that that for the charges we consider the merger condition (4.55) becomes

$$2\,n\,\Delta N = \Delta n\,N\,. \tag{5.7}$$

Using these two equations, we finally have

$$\Delta\mathcal{M} \equiv \widetilde{\mathcal{M}} - \mathcal{M} = n\,\Delta n(3N^2 - 4nJ) \geq 0\,, \tag{5.8}$$

with equality if and only if the original black ring has vanishing horizon area. Hence, for mergers with $\nu = -1$ or $x = \pm 1$, we have proved that chronology is protected, and that the second law of black hole thermodynamics holds. This conclusion is similar to that of [43, 44, 46] for supertube-black hole mergers.

However for $\nu \neq -1$ the situation is rather more subtle. First, the complete supergravity solution is not known for mergers in which the supertube winds around an S^1 in the S^2 of the horizon. As a result we cannot identify the supertube DBI charges with simple supergravity charges. In addition it is not clear how to identify directly the charges carried across the horizon during the merger. If one simply chooses one of the patches discussed above and assumes that the supertube carries its constituent DBI or GH charges across the horizon then the x-dependence in the merger condition (4.49) can lead to mergers in which the horizon area of the black ring decreases, thus contradicting black hole thermodynamics.

The most likely solution to this conundrum is that the charges carried by the supertube across the horizon are not the same as the constituent supertube charges $\overline{N}^{ST}, \overline{J}^{ST}$, but are modified *in an x-dependent way*, so as not to decrease the horizon area. This would imply that in $\nu \neq -1$, $x \neq \pm 1$ mergers the supertube brings in not only its intrinsic charges, but also some of the charge and angular momentum dissolved in supergravity fluxes. Since it is unclear how the dynamics of this charge can be captured via a Born-Infeld analysis, we believe that the understanding of this phenomenon and a resolution of this puzzle will probably come from finding the fully back-reacted supergravity solution corresponding to the $\nu \neq -1$ mergers.[16]

5.2 Mergers of black rings with three-charge two-dipole-charge supertubes

Another interesting example for illustrating chronology protection is the merger of a three-charge two-dipole charge supertube with another supertube of the same kind, that can also be thought of as a singular black ring that has one zero dipole charge $n_3^{BR} = 0$. Such a singular black ring must have vanishing horizon area, and to avoid closed timelike curves it must satisfy the charge condition [56]:

$$n_1^{BR} N_1^{BR} = n_2^{BR} N_2^{BR}\,. \tag{5.9}$$

Similarly, the three-charge supertube considered above has no NS5 dipole charge ($n_3 = 0$) and also satisfies

$$n_1^{ST} N_1^{ST} = n_2^{ST} N_2^{ST}\,. \tag{5.10}$$

[16]Such mergers do not have a tri-holomorphic U(1) invariance and hence the supergravity solution will be more complicated than the solutions with a Gibbons-Hawking base presented here.

Since the merger produces another two-dipole three-charge tube, it must also satisfy the regularity condition:

$$\left(n_1^{BR} + n_1^{ST}\right)\left(N_1^{BR} + N_1^{ST}\right) - \left(n_2^{BR} + n_2^{ST}\right)\left(N_2^{BR} + N_2^{ST}\right) = 0\,, \tag{5.11}$$

which is equivalent to

$$n_1^{BR} N_1^{ST} + n_1^{ST} N_1^{BR} - \left(n_2^{BR} N_2^{ST} + n_2^{ST} N_2^{BR}\right) = 0\,. \tag{5.12}$$

On the other hand, the merger condition (4.68) for $n_3^{BR} = 0$ yields:

$$\left(n_1^{BR} N_1^{ST} + n_2^{BR} N_2^{ST}\right) - \left(n_1^{ST} N_1^{BR} + n_2^{ST} N_2^{BR}\right) = 0\,. \tag{5.13}$$

To establish chronology protection one must show that (5.13) implies (5.12).

However, one also knows that the two merging objects obey (5.9) and (5.10). Multiplying (5.13) by $n_2^{BR} n_2^{ST}$ and using (5.9) and (5.10) one obtains:

$$\left(n_2^{BR} N_1^{ST} - n_2^{ST} N_1^{BR}\right)\left(n_1^{ST}\, n_2^{BR} + n_2^{ST}\, n_1^{BR}\right) = 0\,. \tag{5.14}$$

Similarly, one finds that (5.12) is equivalent to

$$\left(n_2^{BR} N_1^{ST} - n_2^{ST} N_1^{BR}\right)\left(n_1^{ST}\, n_2^{BR} - n_2^{ST}\, n_1^{BR}\right) = 0\,. \tag{5.15}$$

Since all the n's are positive, we see that (5.14) implies (5.15) and so the merger condition (5.13) implies that the regularity condition (5.12) is satisfied. Hence, the merger of two three-charge two-dipole charge supertubes always respects chronology protection.

We can also consider a merger of a three-charge two-dipole charge supertube with a fully fledged black ring, we take for simplicity equal charges and dipoles: $n_1^{BR} = n_2^{BR} = n_3^{BR} = n$, $N_1^{BR} = N_2^{BR} = N_3^{BR} = N$, $N_1^{ST} = N_2^{ST} = N_3^{ST} = \Delta N$ and $n_1^{ST} = n_2^{ST} = \Delta n$. The non-negativity of the initial black ring entropy implies that $3N^2 \geq 4nJ$ and the merger condition[17] becomes $3n\Delta N = 2\Delta n N$. Also remembering that angular momentum of the three-charge supertube is given by

$$J^{ST} = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} = \frac{N_2^{ST} N_3^{ST}}{n_1^{ST}} \tag{5.16}$$

and hence $\Delta J = \Delta N^2/\Delta n$, we obtain

$$\Delta\mathcal{M} \equiv \widetilde{\mathcal{M}} - \mathcal{M} = \frac{4}{9}\left(7N^2 - 9nJ\right)\left(2n\Delta n + \Delta n^2\right)\,. \tag{5.17}$$

Since $N^2 \geq \frac{4}{3}nJ$ this merger is always irreversible, and does not violate chronology protection.

[17] We consider $\nu = -1$ tubes in the $c = -1$ patch; all the subtleties having to do with changing patches are the same as for two-charge supertubes.

6 Fluctuating supertubes and entropy enhancement

This section is devoted to an in-depth review of the Born-Infeld calculation of the entropy coming from the shape modes of supertubes, as well as to an extension of this calculation to a supertube in a black-ring background. This calculation demonstrates that one can equally obtain an enhanced entropy from fluctuations along the compact internal directions of the solution and fluctuations in the non-compact directions of the solution. Furthermore, as we have shown in the previous sections of this paper, we expect the latter supertube fluctuations to give rise to smooth horizonless solutions. Hence, our analysis strongly supports the existence of smooth horizonless three-charge solutions that depend on arbitrary continuous functions, and whose entropy is much larger than their typical charge, and might even be as large as the square root of the cube of their charge. That is, it might be black-hole-like.

Our goal is to quantize the small oscillations about round two-charge supertubes in flat space, black-hole, black-ring, and generic three-charge backgrounds, and to examine the entropy coming from these fluctuations. We find it convenient to work in the D0-D4-F1 duality frame, and our approach follows that of [2, 13] (see also [57]).

We begin by reviewing the Marolf-Palmer entropy calculation for a supertube in flat space, and in the following subsections extend this calculation for a supertube in a 3-charge black hole background and in a black ring background. In the last subsection we also include, for completeness, the entropy calculation in the background of a general solution with a Gibbons-Hawking base space [13].

As first reported in [13], in the latter two backgrounds we find a non-trivial enhancement of the entropy of a supertube when the dipole magnetic fields are large. This enhancement arises because the entropy that can be stored in a supertube is governed not by the electric charges of the supertube (as in flat space or in a black hole background) but by its locally-defined *effective charges*, that can get large contributions from the interactions of the dipole moment of the supertube with the magnetic fluxes of the background.

6.1 Flat space

In the absence of background fluxes, the WZ action of the supertube is zero, and the DBI action (4.7) reduces to

$$S = -T_{D2}\int dt dz d\theta \sqrt{R^2(1-\mathcal{F}_{tz}^2)+\mathcal{F}_{z\theta}^2}\,, \tag{6.1}$$

where R is the radius of the supertube and its embedding is

$$t=\xi^0\,, \qquad z=\xi^1\,, \qquad \varphi_1=\theta\ . \tag{6.2}$$

The charges of the tube are given by (4.12):

$$N_1^{ST}=n_2^{ST}\mathcal{F}_{z\theta}\,, \qquad N_3^{ST}=n_2^{ST}\frac{R^2}{\mathcal{F}_{z\theta}}\,, \tag{6.3}$$

where the factors of n_2^{ST} come from multiple windings in θ. Similarly the radius relation (4.17) reduces to:

$$N_1^{ST} N_3^{ST} = \left(n_2^{ST}\right)^2 R^2 \,. \tag{6.4}$$

The angular momentum of the supertube is (C.12):

$$J = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} = n_2^{ST} R^2 \,. \tag{6.5}$$

The foregoing results apply to round (maximally spinning) supertubes. Supertubes of arbitrary shape will have more complicated expressions for their conserved quantities and will generically have smaller angular momentum.

In this subsection we will perform a simplified version of the analysis in [2], which will be enough to give us the correct dependence of the entropy on the supertube charges. We consider small fluctuations of the supertube in the six directions transverse to its world-volume:

$$x_i \to x_i + \eta_i(t,\theta) \,, \qquad\qquad i = 1,\ldots,6 \,, \tag{6.6}$$

where four of these fluctuations take place on the compact T^4 and the other two are radial coordinates in the non-compact space. In general there are eight independent fluctuation modes for the supertube, consisting of seven transverse coordinate motions and a charge density fluctuation (which also affects the shape). To keep the computations simple here, we have restricted to a representative sample of oscillations in both the compactification space and in the space-time. Since we are only interested in BPS fluctuations we will also restrict η_i to depend only upon t and θ [2].[18]

The effective Lagrangian for the fluctuations is obtained by expanding the DBI Lagrangian of the supertube

$$L_\eta = -T_{D2}\left[(1-\mathcal{F}_{tz}^2 - \dot\eta_i\dot\eta_i)(R^2+\eta_i'\eta_i') - 2\mathcal{F}_{tz}\mathcal{F}_{z\theta}\dot\eta_i\eta_i' + \mathcal{F}_{z\theta}^2(1-\dot\eta_i\dot\eta_i) + (\dot\eta_i\eta_i')^2\right]^{1/2} \,, \tag{6.7}$$

where the repeated index i is summed over. The canonical momenta conjugate to η_i are:

$$\Pi_i = \int_0^{2\pi L_z} dz \, \frac{\partial L_\eta}{\partial \dot\eta_i}\bigg|_{\dot\eta_i=0\,,\,\mathcal{F}_{tz}=1} = \frac{1}{2\pi}\eta_i' \,, \tag{6.8}$$

and the canonical commutation relations are:

$$[\eta_j(t,\theta), \Pi_k(t,\theta')] = i\delta_{jk}\delta(\theta-\theta') \,. \tag{6.9}$$

The BPS modes η_i then can be expanded as:

$$\eta_i = \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right] \tag{6.10}$$

[18] The time dependent modes will break supersymmetry. Hence, we will retain the time dependence of η_i to compute momenta and quantize the system but then we will set $\partial_t\eta_i \equiv \dot\eta_i = 0$.

where $(a_k^i)^\dagger$ and a_k^i are creation and annihilation operators for the $k^{\rm th}$ harmonic. The normalization has been chosen such that:[19]

$$\left[\left(a_k^i\right)^\dagger, a_{k'}^j\right] = \delta^{ij}\delta_{k,k'} \tag{6.11}$$

It is not hard to see that the fluctuations do not change N_1^{ST} and the angular momentum J. The charge N_3^{ST} becomes:

$$N_3^{ST} = \frac{1}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \left.\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\right|_{\mathcal{F}_{tz}=1} = \frac{T_{D2}}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \frac{(R^2+\eta_i'\eta_i')}{\mathcal{F}_{z\theta}}\,, \tag{6.12}$$

from which one finds

$$\sum_{i=1}^{6}\sum_{k>0} k(a_k^i)^\dagger a_k^i = L_z T_{D2}\int_0^{2\pi n_2^{ST}} d\theta \int_0^{2\pi n_2^{ST}} d\theta' \sum_{i=1}^{6}\eta_i'\eta_i' \tag{6.13}$$

$$= N_1^{ST}N_3^{ST} - (n_2^{ST})^2R^2 = N_1^{ST}N_3^{ST} - n_2^{ST}J\,. \tag{6.14}$$

The left hand side of this expression can be thought of as the energy of a system of six massless bosons in $(1+1)$ dimensions. Due to supersymmetry there will also be six corresponding fermionic degrees of freedom. The total central charge of the system is thus $c = 9$, and so the entropy of this system is given by the Cardy formula:

$$S = 2\pi\sqrt{\frac{c}{6}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J} = 2\pi\sqrt{\frac{3}{2}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J}\,. \tag{6.15}$$

If we had included all eight bosonic fluctuation modes then we would have had eight bosons and eight fermions and hence a theory with $c = 12$ and with the entropy:

$$S_{ST} = 2\pi\sqrt{2}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J}\,. \tag{6.16}$$

This is the correct central charge and it yields the correct supertube entropy [2]. By restricting our analysis to six of the shape modes and ignoring the other supersymmetric modes we have obtained a finite, but well understood, fraction of the supertube entropy. Since our purpose here is to analyze when entropy enhancement happens, and when it does not, we will only be interested on the dependence of the supertube entropy on the macroscopic charges, and not pay particular attention to numerical coefficients. Restricting our analysis in more general backgrounds to transverse BPS fluctuations and counting the entropy coming from these modes will therefore be enough to illustrate the physics of entropy enhancement.

[19]Technically, to get this normalization correct we need to include the mode expansion of the non-BPS modes in (6.10). Ignoring the non-BPS modes gives an incorrect factor of $\sqrt{2}$ in the normalization of the η_i. Here we have given the correctly normalized expressions that one would obtain if one included the non-BPS modes.

JHEP07(2009)106

6.2 The three-charge black hole

A two-charge round supertube in the background of a three-charge BPS rotating (BMPV) black hole was discussed in section 4.1. Here we will use the metric and background fields presented in section 4.1 and consider small shape fluctuations in the directions transverse to the world-volume of the supertube. We are again interested only in BPS excitations, which have the following form

$$x_i \to x_i + \eta_i(t,\theta)\,, \quad i = 1,2,3,4\,, \qquad u \to u + \eta_5(t,\theta)\,, \qquad v \to v + \eta_6(t,\theta)\,, \tag{6.17}$$

where we have defined the metric on the four-torus to be

$$ds^2_{T^4} = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\,. \tag{6.18}$$

and the supertube embedding is the same as (6.2). One can use the sum of the DBI and WZ actions, find an effective action for the supertube fluctuations and compute the momenta conjugate to η_5, η_6 and η_i:

$$\Pi_{\eta_5} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_5}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_5'\,, \tag{6.19}$$

$$\Pi_{\eta_6} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_6}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_6'\,, \tag{6.20}$$

$$\Pi_{\eta_i} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_i}\right)\bigg|_{BPS} = \frac{1}{2\pi}\eta_i'\,, \tag{6.21}$$

where the subscript "BPS" means that we have evaluated everything "on shell," which means we have imposed the BPS conditions of no time dependence and $\mathcal{F}_{tz} = 1$.

The BPS modes η_i, η_5 and η_6 then can be expanded as

$$\begin{aligned}
\eta_i &= \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right]\,,\\
\eta_5 &= \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^5)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right]\,,\\
\eta_6 &= \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^6)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right]\,.
\end{aligned} \tag{6.22}$$

At first glance, the physics of the η_i fluctuations along the torus appears very different from that of the fluctuations in the spacetime direction, η_5 and η_6; indeed the latter have a factor of Z_2 in the denominator, and this factor becomes arbitrarily large when the supertube is near the horizon of a black hole.

The charge N_1^{ST} is the same as that of the round supertube, but the charge N_3^{ST} is modified to:

$$N_3^{ST} = \frac{1}{T_{F1}}\int d\theta\, \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{BPS} = \frac{T_{D2}}{T_{F1}\mathcal{F}_{z\theta}}\int d\theta \left(Z_2 u^2 + Z_2 1\left[(\eta_5')^2 + (\eta_6')^2\right] + \sum_{i=1}^{4}(\eta_i')^2\right)\,. \tag{6.23}$$

Using similar arguments to those given for the flat space background one finds the entropy of the BPS shape modes to be:

$$S = 2\pi\sqrt{\frac{3}{2}}\sqrt{N_1^{ST}N_3^{ST} - (n_2^{ST})^2 Z_2 u^2}\,. \tag{6.24}$$

Hence, despite the presence of the warp factor Z_2 in the radius relation and in the mode expansions (6.22), the entropy of the supertube depends on its charges in exactly the same way as in flat space, and hence there is no entropy enhancement.

6.3 The three-charge black ring background

We now consider small shape fluctuations around the round supertube in a black ring background presented in section 4.2. The important new element is that this background has non-zero magnetic dipole charges and these will enter the calculation in some very non-trivial ways.

Again we consider the fluctuations (6.17) and use the DBI and WZ actions to find an effective action for the fluctuations. After straightforward calculations on can compute the momenta conjugate to η_5, η_6 and η_i:

$$\Pi_{\eta_5} = \int dz \left(\frac{\partial\mathcal{L}}{\partial\dot\eta_5}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\frac{R^2}{(y^2-1)(x-y)^2}\eta_5'\,, \tag{6.25}$$

$$\Pi_{\eta_6} = \int dz \left(\frac{\partial\mathcal{L}}{\partial\dot\eta_6}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\frac{R^2}{(1-x^2)(x-y)^2}\eta_6'\,, \tag{6.26}$$

$$\Pi_{\eta_i} = \int dz \left(\frac{\partial\mathcal{L}}{\partial\dot\eta_i}\right)\bigg|_{BPS} = \frac{1}{2\pi}\eta_i'\,, \tag{6.27}$$

The BPS modes η_i, η_5 and η_6 can be expanded as:

$$\eta_i = \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^i)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right],$$

$$\eta_5 = \sqrt{\frac{(y^2-1)(x-y)^2}{2Z_2R^2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^5)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right], \tag{6.28}$$

$$\eta_6 = \sqrt{\frac{(1-x^2)(x-y)^2}{2Z_2R^2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^6)^\dagger}{\sqrt{|k|}} + \text{h.c.}\right].$$

Suppose that we have a round supertube parallel to the ring ($t = \xi^0,\ z = \xi^1,\ \varphi_1 = -\theta$), then for the F1 charge of the supertube one finds

$$N_3^{ST} = \frac{1}{T_{F1}}\int_0^{2\pi n_2^{ST}}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\right)\bigg|_{BPS} \tag{6.29}$$

$$= \frac{T_{D2}}{T_{F1}}n_2^{ST}n_1(1+y) + \frac{T_{D2}}{T_{F1}(\mathcal{F}_{z\theta} - \frac{n_3}{2}(1+y))}\left[\frac{Z_2R^2(y^2-1)}{(x-y)^2}\right. \tag{6.30}$$

$$\left. + Z_2\frac{R^2}{(y^2-1)(x-y)^2}(\eta_5')^2 + Z_4\frac{R^2}{(1-x^2)(x-y)^2}(\eta_6')^2 + (\eta_i'\eta_i')\right]. \tag{6.31}$$

The expression for the entropy coming from the shape oscillations now becomes:

$$S = 2\pi\sqrt{\frac{3}{2}}\left\{\left[N_1^{ST} - \frac{1}{2}n_2^{ST}n_3(1+y)\right]\left[N_3^{ST} - \frac{1}{2}n_2^{ST}n_1(1+y)\right] - (n_2^{ST})^2\frac{Z_2R^2(y^2-1)}{(x-y)^2}\right\}^{\frac{1}{2}} \tag{6.32}$$

Note that for a supertube located near the black ring ($y \to -\infty$) one has a huge entropy enhancement due to the dipole-dipole interaction.

For completeness, it is equally easy to consider a round supertube orthogonal to the black ring ($t = \xi^0$, $z = \xi^1$, $\varphi_2 = -\theta$). One then finds that the entropy of the shape modes is:

$$S = 2\pi\sqrt{\frac{3}{2}}\left\{\left[N_1^{ST} + \frac{1}{2}n_2^{ST}n_3(x+c)\right]\left[N_3^{ST} + \frac{1}{2}n_2^{ST}n_1(x+c)\right] - (n_2^{ST})^2\frac{Z_2R^2(1-x^2)}{(x-y)^2}\right\}^{\frac{1}{2}} \tag{6.33}$$

While there is still a dipole-dipole interaction, the entropy enhancement does not grow arbitrarily large because the coordinate x has a finite range ($x \in (-1,1)$).

6.4 Solution with a general Gibbons-Hawking base

For the sake of completeness, it is worth reviewing also the entropy enhancement for a supertube in a three-charge background with a Gibbons-Hawking base. For this background, one can only calculate easily the entropy coming from the internal fluctuations of the supertube. The entropy coming from fluctuations of the supertube in the spacetime directions is more complicated than for the black ring background.

For this background the supertube action becomes:

$$\begin{aligned} S = T_{D2}\int d^3\xi\Bigg\{&\left[\left(\frac{1}{Z_1}-1\right)\mathcal{F}_{z\theta} + \frac{K^3}{Z_1V} + \left(\frac{\mu}{Z_1} - \frac{K^1}{V}\right)(\mathcal{F}_{tz}-1)\right] \\ &-\left[\frac{1}{V^2Z_1^2}\left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + VZ_1Z_2(1-\mathcal{F}_{tz})(2-Z_3(1-\mathcal{F}_{tz}))\right]\right]^{1/2}\Bigg\}. \end{aligned} \tag{6.34}$$

Because of the complexity of this background, we consider small shape oscillations in the compactification manifold, T^4, around a round supertube along the GH fiber :

$$t = \xi^0\,, \qquad z = \xi^1\,, \qquad \psi = \theta\,, \qquad x_i \to x_i + \eta_i(t,\theta) \qquad i = 1,2,3,4\,. \tag{6.35}$$

The quantization proceeds exactly as before and the conserved electric charges are now:

$$\begin{aligned} N_1^{ST} &= \frac{T_{D2}}{T_{D0}}\int_0^{2\pi L_z} dz \int_0^{2\pi n_2^{ST}} d\theta\, \mathcal{F}_{z\theta} = n_2^{ST}\mathcal{F}_{z\theta}\,, \\ N_3^{ST} &= \frac{T_{D2}}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta\left[-\frac{K^1}{V} + \frac{1}{\mathcal{F}_{z\theta}+V^{-1}K^3}\left(\frac{Z_2}{V} + \sum_i^4(\eta_i')^2\right)\right]. \end{aligned} \tag{6.36}$$

Substituting (6.28) into (6.36) and rearranging using (6.36) leads to:

$$\begin{aligned} \sum_{i=1}^4\sum_{k>0} k(a_k^i)^\dagger a_k^i &= L_zT_{D2}\int_0^{2\pi n_2^{ST}} d\theta \int_0^{2\pi n_2^{ST}} d\theta \sum_{i=1}^4 \eta_i'\eta_i' \\ &= \left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + n_2^{ST}\frac{K^1}{V}\right] - (n_2^{ST})^2\frac{Z_2}{V}\,. \end{aligned} \tag{6.37}$$

and this leads to the following expression for the entropy:

$$S = 2\pi \sqrt{\left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + n_2^{ST}\frac{K^1}{V}\right] - (n_2^{ST})^2\frac{Z_2}{V}}\,. \tag{6.38}$$

6.5 Comments on the supertube effective charges

As we have seen, in flat space and in a BMPV black hole background, the entropy of the two-charge supertube, when expressed in terms of its charges, is simply

$$S \sim \sqrt{Q_1\, Q_3 - J}\,. \tag{6.39}$$

However, if the background has non-trivial dipole magnetic fields the entropy is given by equations (6.33) and (6.38), and can be written as:

$$S \sim \sqrt{Q_1^{eff}\, Q_3^{eff} - J^{eff}}\,. \tag{6.40}$$

Here the *effective charges*, Q_I^{eff} and J^{eff}, involve a non-trivial interaction between the dipoles of the supertube and the dipoles of the background. These effective charges can become arbitrarily large if the supertube moves suitably close to the background dipole sources.

From the perspective of the supertube DBI-WZ action, these effective charges are:

$$Q_1^{eff} \equiv Q_1^{ST} + n_2^{ST}\,\tilde{\xi}^{(1)}\,, \qquad Q_3^{eff} \equiv Q_3^{ST} + n_2^{ST}\,\tilde{\xi}^{(2)}\,, \tag{6.41}$$

where the $\xi^{(I)}$ are defined in (2.9) and $\tilde{\xi}^{(I)}$ denotes the pull-back onto the supertube. There is another way to think about these effective charges when considering the fully back-reacted solution found for a round supertube in GH backgrounds [13] — they give the leading divergence of the warp factors Z_I near the supertube:

$$Q_I^{eff} \equiv 4 \lim_{r_N \to 0} r_N\, Z_I\,, \qquad I = 1, 3\,, \tag{6.42}$$

where the supertube is located at $r_N = 0$. Nicely enough, even if the DBI-WZ action of the supertube is perturbative, it does capture these effective charges via the pull back in (6.41).

As discussed in [13], the crucial insight coming from this analysis is that the entropy of the supertube is not determined in terms of its asymptotic charges (measured at infinity) but in terms of its local effective charges, which depend on the location of the supertube. Hence, the entropy can become very large when the magnetic fields are very strong — this happens for example when the supertube is near the horizon of a black ring, or when it is in a deep scaling horizonless solution [37, 59–61].

Our analysis also demonstrates that entropy enhancement affects *both* the fluctuations of the supertube in the internal (torus) directions, as well as the fluctuations of the supertube in the non-compact transverse space. In a general three-charge background the latter are very hard to analyze, as the non-trivial magnetic field mixes the fluctuation modes. However, in a black ring background this mixing is not present for the supertube fluctuations in the plane transverse to the ring. Our calculation shows that these fluctuations exhibit *the same* amount of entropy enhancement as the torus fluctuations, and hence indicate that entropy enhancement is a feature of all the supertube modes, and not just some. It would be interesting to calculate whether, in a general background, some modes are more enhanced than others, as this would indicate whether the typical microstates of "enhanced" fluctuating supertubes are smooth in supergravity or not. It will be also very interesting to extend our analysis to non-supersymmetric gravity solutions [62, 63].

7 Conclusions

Our purpose in this paper has been four-fold:

First, we proved that if one takes supertubes that are solutions of the Born-Infeld action to a regime of parameters where their back-reaction is important, the fully back-reacted supergravity solution is smooth in the duality frame where the supertubes have D1 and D5 electric charges. The two conditions necessary for the supergravity solution to be free of closed timelike curves and to be smooth are reproduced exactly by the Born-Infeld analysis.

Our analysis strengthens the case for the existence of families of supergravity solutions that have the same charges as black holes, and that depend on arbitrary continuous functions (and hence have a moduli space of infinite dimension). Furthermore, these solutions are smooth and horizonless in the regime of parameters in which the corresponding black hole has a macroscopic horizon.

The second purpose of the paper has been to identify the relation between the charges of supertubes and black rings that appear in the exact supergravity description, and those that appear in the microscopic (Born Infeld) description.

We have seen in section 4.7 that a given five-dimensional black ring can be embedded in Taub-NUT in two ways, that differ from each other by the choice of the location of the Dirac string in the gauge potentials. One can furthermore find black ring embeddings with multiple Dirac strings, that depend on several parameters, and these can be related to each other by gauge transformations. The Gibbons-Hawking charges of the black ring, which give the electrical charges of the corresponding four-dimensional black hole, are different in different patches (4.37), (4.38). Nevertheless, the $E_{7(7)}$ quartic that gives the microscopic entropy of the black ring, is independent of the choice of patch.

It is interesting to note that the entropy of extremal non-BPS black rings has been recently expressed in terms of the $E_{7(7)}$ quartic invariant as a function of the asymptotic charges, and a certain angular momentum parameter J [55]. Our analysis establishes that the apparent four-dimensional charges (that appear in this invariant) depend on the location of the Dirac string, and that one can switch between the asymptotic charges of the ring and the intrinsic charges by a gauge transformation. This transformation nevertheless also changes the angular momentum parameter, and thus the question that should be asked in trying to find the microscopic description of extremal non-BPS black rings is not "Why does a certain charge appear in the quartic invariant?" but rather "Why, for a given choice of charges, does a certain angular momentum parameter appear in the quartic invariant?"

We have also found the relation between the charges of supertubes that appear in their Born-Infeld description, and those that appear in their supergravity description. We have established that if a supertube that gives rise to a solution with a Gibbons-Hawking base is put at a smooth location,[20] its Born-Infeld electric charges are equal to the Gibbons-Hawking charges of the supergravity solution. Since the Gibbons-Hawking charges are the ones that contributes to the asymptotic charge of a solution, and since these charges are much smaller than the enhanced charges (that give the supertube entropy in a three-charge background) our analysis definitively establishes the phenomenon of entropy enhancement: a given two-charge supertube in a three-charge two-dipole charge background has an entropy much larger than one would expect from the amount of charge visible from infinity.

The third aim of our paper has been to analyze issues related to black-hole thermodynamics and chronology protection when a supertube is merged with a black ring. If supertubes respect the triholomrphic U(1) isometry of the ring, and are able to merge with a black ring, then this neither decreases the ring entropy nor creates closed timelike curves. The supertubes that might do this, and hence are "dangerous" for chronology protection and thermodynamics, are unable to merge with the ring.

The situation is a bit more subtle with supertubes that *do not* respect the triholomorphic U(1) isometry of the ring, and wind around S^1 latitude circles in the S^2 of the black ring horizon. We have found that if the charge these supertubes carry into a black ring is given by their Born-Infeld charge, then chronology protection and black-hole thermodynamics can be violated! The only way these are not violated is if the charge brought into the black ring depends continuously on the angle at which the supertube merges with the ring (which is the angle of the S^1 latitude circle it wraps). It would be interesting to understand the origin of this very puzzling fact, by constructed the fully back-reacted solution corresponding to this merger. This solution will have a U(1) isometry, but not a triholomorphic one, and will hence not be a Gibbons-Hawking solution, but a more general one of the type constructed in [41, 46, 64].

The fourth aim of the paper was to extend the entropy enhancement calculation of [13] to supertubes that oscillate both in the internal compact directions and in spacetime non-compact directions. Such a calculation is generically quite complicated: if a solution depends on these directions, this mixes the corresponding oscillator modes of the supertube, which makes the counting much more involved. Nevertheless, we have found a class of examples in which this mixing is not present, and the calculation of the entropy coming from the spacetime modes of the supertube is as simple as that coming from the internal modes.

Our results show that the two kind of modes contribute to the enhanced entropy equally, despite the presence of different (large) factors in the mode expansions. If, as we expect, the entropy coming from these fluctuations will be black-hole-like, and therefore the fluctuating supertubes will give the typical microstates of the corresponding black hole, these microstates will have a non-trivial transverse size, and the smooth horizonless microstates will act as representatives for all the black hole microstates [13, 18].

The obvious question left unanswered by our analysis is what is the enhanced entropy coming from the modes that mix. This question requires a more tedious analysis than we have done, but its answer could have dramatic consequences. If this enhanced entropy is

[20] More precisely, not exactly on top of a Dirac string.

JHEP07(2009)106

equal or less than that coming from the internal modes, then most likely the typical black-hole microstate geometries will be given by a combination of internal and transverse space oscillations, which in general will not be smooth (but may have smooth representatives). However, if the entropy coming from the transverse modes that mix is greater than the one coming from the internal directions, then the typical microstates might all be given by smooth horiozonless supergravity solutions.

To recapitulate, we have proven that the supergravity and the Born Infeld descriptions of supertube agree, found the four-dimensional charges of five-dimensional black rings and supertubes, analyzed chronology protection and black hole thermodynamics during black-ring supertube mergers, and established that the entropies of supertube modes in the internal directions in the spacetime directions are enhanced equally, and hence these modes contribute equally to the entropy of the supertube.

We have also filled in a few details in the analysis of supertubes and black rings solutions: we have dualized the black ring and the more general multi-center solutions with a Gibbons-Hawking base to various duality frames (in appendix A), and have found (to our knowledge for the first time) the exact form of the magnetic potentials in these solutions. We have also calculated (in appendix C) the angular momenta of a supertube of arbitrary shape in a general solution with an $\mathbb{R}^4$ base, and shown that the contribution of a piece of an arbitrarily-shaped supertube to the angular momentum along the direction of this piece is the same as for a piece of a circular supertube, and is in fact a universal quantity, as suggested also by the supergravity analysis.

Last, but not least, we have shown (in appendix B) that all the three-charge, three-dipole charge solutions with a Gibbons-Hawking base constructed so far can be dualized to the duality frame where they have D1, D5 and momentum charges, and can be scaled[21] in such a way as to become asymptotically $AdS_3 \times S^3$. Hence all these smooth horizonless solutions are dual via the AdS/CFT correspondence to microstates of the D1-D5 CFT. It would be very interesting to extend the holographic methods of [19] (that were successfully used in [7] for two-charge microstates) to the analysis of these three-charge geometries. This would enable one to establish whether the geometries constructed so far are dual to typical CFT microstates, whether the geometries dual to these microstates have Planck-scale curvature or are well-described in supergravity, and whether the smooth microstate geometries constructed so far can act as representatives of the typical microstates.

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[21] This has been done before for black rings [15], but not for general multi-center solutions.

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A Three charge solutions and T-duality

A.1 T-duality transformations

In this appendix we summarize the T-duality transformation rules for type II theories with non-zero RR fields. These rules are derived in [65] and can be considered a generalization of the Buscher rules [66]. In the expressions below we will adopt the conventions and notation of [67], the different RR forms are denoted with $C^{(n)}$ and the fields obtained after the T-duality transformations are denoted with a tilde, $w = x_9$ is the M-theory compactification direction and x is the T-duality direction.

The set of fields in the low energy limit of M-theory, i.e. eleven-dimensional supergravity, are:

$$G_{\mu\nu} \qquad \text{and} \qquad A_{\mu\nu\rho}\,. \tag{A.1}$$

After the compactification along $w = x_9$ we are left with type IIA supergravity with the fields

$$g_{\mu\nu}, \qquad C^{(3)}_{\mu\nu\rho}, \qquad B_{\mu\nu}, \qquad C^{(1)}_{\mu}, \qquad \Phi\,, \tag{A.2}$$

which are related to the eleven-dimensional fields as follows (note that we are working in string frame):

$$\begin{aligned} g_{\mu\nu} &= \sqrt{G_{ww}}\left(G_{\mu\nu} + \frac{G_{\mu w}G_{\nu w}}{G_{ww}}\right), & & & C^{(1)}_{\mu} &= \frac{G_{\mu w}}{G_{ww}}\,, \\ C^{(3)}_{\mu\nu\rho} &= A_{\mu\nu\rho}\,, & B_{\mu\nu} &= A_{\mu\nu w}\,, & \Phi &= \frac{3}{4}\log(G_{ww})\,. \end{aligned} \tag{A.3}$$

The type IIB fields are:

$$g_{\mu\nu}, \qquad B_{\mu\nu}, \qquad \Phi, \qquad C^{(0)}, \qquad C^{(2)}_{\mu\nu}, \qquad C^{(4)}_{\mu\nu\rho\sigma}\,. \tag{A.4}$$

The T-duality rules for the metric and the NS-NS fields are:

$$\begin{aligned} \tilde{g}_{xx} &= \frac{1}{g_{xx}}, & \tilde{g}_{\mu x} &= \frac{B_{\mu x}}{g_{xx}}, & \tilde{g}_{\mu\nu} &= g_{\mu\nu} - \frac{g_{\mu x}g_{\nu x} - B_{\mu x}B_{\nu x}}{g_{xx}}\,, \\ \tilde{B}_{\mu x} &= \frac{g_{\mu x}}{g_{xx}}, & \tilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu x}g_{\nu x} - g_{\mu x}B_{\nu x}}{g_{xx}}\,, & \tilde{\Phi} &= \Phi - \frac{1}{2}\log g_{xx}\,. \end{aligned} \tag{A.5}$$

The RR forms transform under T-duality as:

$$\begin{aligned} \tilde{C}^{(n)}_{\mu\ldots\nu\alpha x} &= C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|x}g_{|\alpha]x}}{g_{xx}}\,, \\ \tilde{C}^{(n)}_{\mu\ldots\nu\alpha\beta} &= C^{(n+1)}_{\mu\ldots\nu\alpha\beta x} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}B_{\beta]x} + n(n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|x}B_{|\alpha|x}g_{|\beta]x}}{g_{xx}}\,. \end{aligned} \tag{A.6}$$

	0	1	2	3	4	5	6	7	8	9	10
(D1) M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
(D5) M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$
(P) M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
(d5) M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
(d1) M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
(kkm) M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$

Table 1. The configuration of branes in M-theory that preserves the four supersymmetries of the M2-M2-M2 three-charge black hole [20]. The vertical arrows represent the directions along which the branes are extended and the horizontal arrows represent smearing directions. The functions $y^\mu(\phi)$ describe a closed curve which is wrapped by the M5 branes. In the first column we have indicated also the brane identification in the D1-D5-P duality frame.

Alternatively one can transform the RR field strengths as follows (for a detailed derivation of these rules see appendix A of [7])

$$\begin{aligned}\tilde{F}^{(n)}_{\mu_1\ldots\mu_{n-1}x} &= F^{(n-1)}_{\mu_1\ldots\mu_{n-1}} + (n-1)(-1)^n \frac{g_{x[\mu_1}F^{(n-1)}_{\mu_2\ldots\mu_{n-1}]x}}{g_{xx}}\,,\\ \tilde{F}^{(n)}_{\mu_1\ldots\mu_n} &= F^{(n+1)}_{\mu_1\ldots\mu_n x} - n(-1)^n B_{x[\mu_1}F^{(n-1)}_{\mu_2\ldots\mu_n]} - n(n-1)\frac{B_{x[\mu_1}g_{\mu_2|x|}F^{(n-1)}_{\mu_3..\mu_n]x}}{g_{xx}}\,.\end{aligned} \tag{A.7}$$

We now give the explicit transformations that take us from the M-theory duality frame in section 2, to solutions in other useful duality frames. In table 1 we specify the directions along which the M2 branes and the M5 branes are wrapped or smeared.

A.2 Three charge solutions in different duality frames

Compactification along x_9. The first step is to compactify the eleven-dimensional solution, presented in section 2, along x_9, in this way we obtain the following combination of "electric"[22]

$$N_1 :\; D2\,(56) \qquad N_2 :\; D2\,(78) \qquad N_3 :\; F1\,(z) \tag{A.8}$$

and "dipole" branes

$$n_1 :\; D4\,(y78z) \qquad n_2 :\; D4\,(y56z) \qquad n_3 :\; NS5\,(y5678) \tag{A.9}$$

in Type IIA. From now on we will denote $x_{10} = z$. The ten-dimensional *string frame* metric is

$$ds^2_{10} = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2 + \sqrt{Z_1Z_2}ds^2_4 + \frac{\sqrt{Z_1Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_2}{Z_1}}(dx_5^2+dx_6^2) + \sqrt{\frac{Z_1}{Z_2}}(dx_7^2+dx_8^2) \tag{A.10}$$

[22]We are choosing x_9 to be the M-theory circle in order to match the conventions in the literature for the global signs of the B-field and the RR potentials for the BMPV black hole [47] and the supersymmetric black ring solutions [42].

The dilaton and the Kalb-Ramond field are

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1 Z_2}{Z_3^2}\right), \qquad\qquad B = -A^{(3)} \wedge dz\,. \tag{A.11}$$

The RR ("electric") forms are

$$C^{(1)} = 0\,, \qquad\qquad C^{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8\,, \tag{A.12}$$

and the four-form field strength is[23]

$$\begin{aligned}\widetilde{F}^{(4)} &= dC^{(3)} + dB \wedge C^{(1)} = A^{(1)} \wedge dx_5 \wedge dx_6 + dA^{(2)} \wedge dx_7 \wedge dx_8 &\text{(A.13)}\\ &= d\mathcal{F}^{(1)} \wedge dx_5 \wedge dx_6 + \mathcal{F}^{(2)} \wedge dx_7 \wedge dx_8\,, &\text{(A.14)}\end{aligned}$$

where we have used the notation $\mathcal{F}^{(I)} = dA^{(I)}$. Now we will perform a chain of T-dualities in order to arrive at the desired frame.

T-duality along x_5. A T-duality along the x_5 direction brings us to Type IIB with the following sets of "electric"

$$N_1 :\ D1\,(6) \qquad N_2 :\ D3\,(578) \qquad N_3 :\ F1\,(z) \tag{A.15}$$

and "dipole" branes

$$n_1 :\ D5\,(y578z) \qquad n_2 :\ D3\,(y6z) \qquad n_3 :\ NS5\,(y5678)\,. \tag{A.16}$$

The metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_2}{Z_1}}dx_6^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_7^2 + dx_8^2)\,. \tag{A.17}$$

The other NS-NS fields are

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^2}{Z_3^2}\right), \qquad\qquad B = -A^{(3)} \wedge dz\,. \tag{A.18}$$

The RR field strengths are

$$\begin{aligned}F^{(3)} &= -\mathcal{F}^{(1)} \wedge dx_6\,,\\ \widetilde{F}^{(5)} &= \mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8 + \star_{10}(\mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8)\,,\end{aligned} \tag{A.19}$$

where in the expression for $\widetilde{F}^{(5)}$ we have added the Hodge dual piece by hand to ensure self-duality [69]. Note that if one is working in the "democratic formalism" (i.e. with both electric and magnetic field strengths) $\widetilde{F}^{(5)}$ will be automatically self-dual, however since we have chosen to T-dualize explicitly only the electric field strengths we have to add the self-dual piece by hand whenever we encounter a five-form field strength after T-dualizing a four-form field strength.

[23] Note that we are using the notation of [68] $\widetilde{F}^{(4)} = dC^{(3)} + dB \wedge C^{(1)}$.

Using the form of the ten-dimensional metric (A.17) one can show that

$$\star_{10}\left(dA^{(2)}\wedge dx_5\wedge dx_7\wedge dx_8\right)=-\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4}\star_5\left(dA^{(2)}\wedge dz\wedge dx_6\right), \tag{A.20}$$

where $\star_5$ is the Hodge dual on the five-dimensional subspace given by the metric

$$ds_5^2=-\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2+\sqrt{Z_1Z_2}ds_4^2\,. \tag{A.21}$$

T-duality along x_6. Now perform T-duality along x_6 to get

$$N_1:\ D0 \qquad N_2:\ D4\,(5678) \qquad N_3:\ F1\,(z) \tag{A.22}$$

"electric"

$$n_1:\ D6\,(y5678z) \qquad n_2:\ D2\,(yz) \qquad n_3:\ NS5\,(y5678) \tag{A.23}$$

and "dipole" branes in Type IIA. The metric is

$$ds_{10}^2=-\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2+\sqrt{Z_1Z_2}ds_4^2+\frac{\sqrt{Z_1Z_2}}{Z_3}dz^2+\sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \tag{A.24}$$

The dilaton and the Kalb-Ramond fields are

$$\Phi=\frac{1}{4}\log\left(\frac{Z_1^3}{Z_2Z_3^2}\right), \qquad B=-A^{(3)}\wedge dz\,. \tag{A.25}$$

The RR field strengths are

$$F^{(2)}=-\mathcal{F}^{(1)}, \qquad \widetilde{F}^{(4)}=-\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4}\star_5\left(\mathcal{F}^{(2)}\right)\wedge dz\,. \tag{A.26}$$

Since we are interested in studying probe two charge supertubes in this background, we will also need the RR potentials since they enter the Wess-Zumino action of the supertube.

Finding the RR and NS-NS potentials in the D0-D4-F1 frame. If everything is consistent, then the Bianchi identities for the field strengths should be satisfied. For the solution given by (2.20)–(2.22), the non-trivial Bianchi identity is:[24]

$$d\widetilde{F}^{(4)}=-F^{(2)}\wedge dB\,. \tag{A.27}$$

Indeed we can use the BPS equations to show that

$$\begin{aligned} d\widetilde{F}^{(4)} &= -d\left(\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4}\star_5\left(\mathcal{F}^{(2)}\right)\right)\wedge dz \\ &= -\left[d\left(\frac{1}{Z_1Z_3}\right)\wedge dk\wedge(dt+k)-d\left(\frac{(dt+k)}{Z_1}\right)\wedge\Theta^3\right. \\ &\qquad \left.-d\left(\frac{(dt+k)}{Z_3}\right)\wedge\Theta^1+\Theta^3\wedge\Theta^1\right]\wedge dz\,. \end{aligned} \tag{A.28}$$

[24] See [68] p. 86.

On the other hand

$$
\begin{aligned}
F^{(2)} \wedge dB &= dA^{(1)} \wedge dA^{(3)} \wedge dz \\
&= \Bigg[d\Big(\frac{1}{Z_1 Z_3}\Big) \wedge dk \wedge (dt+k) - d\Big(\frac{(dt+k)}{Z_1}\Big) \wedge \Theta^3 \\
&\qquad -d\Big(\frac{(dt+k)}{Z_3}\Big) \wedge \Theta^1 + \Theta^3 \wedge \Theta^1 \Bigg] \wedge dz\,.
\end{aligned} \tag{A.29}
$$

So the Bianchi identity is obeyed and it can be checked in a similar manner that the equations of motion of type IIA supergravity are obeyed. Thus confirms the consistency of our calculations.

We will now find the RR three-form potential $C^{(3)}$ in the same duality frame. It satisfies the following differential equation

$$
dC^{(3)} \equiv \tilde{F}^{(4)} + C^{(1)} \wedge H^{(3)}\,. \tag{A.30}
$$

Note that this depends upon a gauge choice for $C^{(1)}$, we choose a gauge in which $C^{(1)}$ is vanishing at asymptotic infinity, namely[25]

$$
C^{(1)} = -A^1 - dt\,. \tag{A.31}
$$

Computing explicitly one finds

$$
dC^{(3)} = \big[\big(-\star_4 dZ_2 + B^{(1)} \wedge \Theta^{(3)}\big) - d\big(Z_3^{-1}(dt+k) \wedge B^{(1)} + dt \wedge A^{(3)}\big)\big] \wedge dx_5\,, \tag{A.32}
$$

and hence

$$
C^{(3)} = -\big(\gamma + Z_3^{-1}(dt+k) \wedge B^{(1)} + dt \wedge A^{(3)}\big) \wedge dx_5\,, \tag{A.33}
$$

where

$$
d\gamma = \big(\star_4\, dZ_2 - B^{(1)} \wedge \Theta^{(3)}\big)\,. \tag{A.34}
$$

So the calculation boils down to integrating for the 2-form γ. Up to this stage we have not assumed any particular form of the four-dimensional base space. If this space is Gibbons-Hawking then the equation for γ can be integrated explicitly. Using the BPS supergravity solutions presented in section 2 it is not hard to show that

$$
\begin{aligned}
\star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)} &= \big(-\partial_a Z_2 + K^1 \partial_a (V^{-1}K^3)\big)\, \frac{1}{2}\epsilon_{abc}(d\psi + A) \wedge dy^b \wedge dy^c \\
&\quad - \xi^{(1)}_a \,\big(\partial_b (V^{-1}K^3)\big)(d\psi + A) \wedge dy^a \wedge dy^b \\
&\quad + V\big(\vec{\xi}^{(1)} \cdot \vec{\nabla}(V^{-1}K^3)\big)\, dy^1 \wedge dy^2 \wedge dy^3\,.
\end{aligned} \tag{A.35}
$$

Recall that $Z_2 = L_2 + V^{-1}K^1K^3$ and define $\vec{\zeta}$ by:

$$
\vec{\nabla} \times \vec{\zeta} \equiv -\vec{\nabla} L_2\,, \tag{A.36}
$$

then using

$$
\Omega^{(a)}_{\pm} = \hat{e}^1 \wedge \hat{e}^{a+1} \pm \frac{1}{2}\epsilon_{abc}\hat{e}^{b+1} \wedge \hat{e}^{c+1}\,, \tag{A.37}
$$

[25] We have fixed $Z_I \sim 1 + \mathcal{O}(r^{-1})$.

one can show that:

$$\star_4\, dZ_2 - B^{(1)} \wedge \Theta^{(3)} = d\big[\big(-\zeta_a - V^{-1}K^3\xi_a^{(1)}\big)\Omega_-^{(a)}\big] \\ -\big(V\,\vec{\nabla}\cdot\vec{\zeta} + K^3\,\vec{\nabla}\cdot\vec{\xi}^{(1)}\big)\, dy^1 \wedge dy^2 \wedge dy^3\,. \tag{A.38}$$

The last term is a multiple of the volume form on $\mathbb{R}^3$ and so is necessarily exact, however, it can be simplified if we chose a gauge for $\vec{\xi}^{(1)}$ and $\vec{\zeta}$:

$$\vec{\nabla}\cdot\vec{\zeta} = \vec{\nabla}\cdot\vec{\xi}^{(1)} = 0\,. \tag{A.39}$$

Then one has:

$$\gamma = -\big[\big(\zeta_a + V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)}\big]\,. \tag{A.40}$$

Finally, let $\vec{r}_i = (y_1 - a_i, y_2 - b_i, y_3 - c_i)$ and let $F \equiv \frac{1}{r_i}$ and then define $\vec{w}$ by $\vec{\nabla}\times\vec{w} \equiv -\vec{\nabla}F$, then the standard solution for $\vec{w}$ is:

$$w = -\frac{y_3 - c_i}{r_i}\,\frac{(y_1 - a_i)\,dy_2 - (y_2 - b_i)\,dy_1}{((y_1 - a_i)^2 + (y_2 - b_i)^2)}\,. \tag{A.41}$$

It is elementary to verify that $\vec{\nabla}\cdot\vec{w} = 0$ and so this is the requisite gauge. Finally the explicit form of the RR three-form potential for a solution with GH base in the D0-D4-F1 frame is

$$C^{(3)} = \big(\zeta_a + V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)} \wedge dz - \big(Z_3^{-1}(dt + k)\wedge B^{(1)} + dt \wedge A^{(3)}\big)\wedge dz\,. \tag{A.42}$$

T-duality along z. Another T-duality along z transforms the system into D1-D5-P frame with

$$N_1 :\ D1\,(z) \qquad N_2 :\ D5\,(5678z) \qquad N_3 :\ P\,(z) \tag{A.43}$$

"electric"

$$n_1 :\ D5\,(y5678) \qquad n_2 :\ D1\,(y) \qquad n_3 :\ kkm\,(y5678z) \tag{A.44}$$

and "dipole" branes. The metric is

$$ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2 + \sqrt{Z_1Z_2}ds_4^2 + \frac{Z_3}{\sqrt{Z_1Z_2}}(dz+A^3)^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \tag{A.45}$$

The dilaton and the Kalb-Ramond field are:

$$\Phi = \frac{1}{2}\log\left(\frac{Z_1}{Z_2}\right), \qquad B = 0\,. \tag{A.46}$$

The RR three-form field strength (it is the only non-zero field strength) is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4} \star_5(\mathcal{F}^{(2)}) - \mathcal{F}^{(1)} \wedge (dz - A^{(3)})\,. \tag{A.47}$$

For the supersymmetric black ring solution in D1-D5-P frame then our general result agrees (up to conventions) with [42]. We can also easily find the RR 2-form potential by T-dualizing (A.42)

$$C^{(2)} = \left(\zeta_a + V^{-1}K^3\xi_a^{(1)}\right)\,\Omega_-^{(a)} - \left(Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\right) \\ + A^{(1)} \wedge (A^{(3)} - dz - dt) + dt \wedge (A^3 - dz)\,. \tag{A.48}$$

B BPS solutions in D1-D5-P frame and their decoupling limit

In this appendix we consider the decoupling limit of the three-charge metric in the D1-D5-P duality frame (A.45). As shown in [15, 42], for a supersymmetric black ring, such a limit takes an asymptotically-flat solution into a solution that is asymptotically $AdS_3 \times S^3 \times T^4$, and is thus dual to a state or an ensemble of states in the D1-D5 CFT.

Like for three-charge black holes and black rings, one can take this limit by sending $\alpha' \to 0$ and scaling the coordinates and the parameters of the solution in such a way that the type IIB metric scales as α'. At this point it is useful to give the form of the "electric" charges Q_I in terms of the parameters of the eleven-dimensional solution:

$$Q_I = -2C_{IJK} \sum_{j=1}^{N} \frac{\tilde{k}_j^J \tilde{k}_j^K}{q_j} \qquad \text{where} \qquad \tilde{k}_j^I = k_j^I - q_j \sum_{i=1}^{N} k_i^I \,. \tag{B.1}$$

The angular momenta are obtained by expanding the one-form k at infinity and one finds:

$$J_R \equiv J_1 + J_2 = C_{IJK} \sum_{j=1}^{N} \frac{\tilde{k}_j^I \tilde{k}_j^J \tilde{k}_j^K}{q_j^2}, \qquad\qquad J_L = J_1 - J_2 = 8\Big|\sum_{j=1}^{N}\sum_{I=1}^{3} \tilde{k}_j^I \vec{y}^{(j)}\Big|, \tag{B.2}$$

where the $\vec{y}^{(j)}$ are the positions of the GH centers. The scaling with α' of the coordinates is the same as for the black hole solution

$$y_1 \sim y_2 \sim y_3 \sim (\alpha')^2 \,, \qquad x_a \sim (\alpha')^{1/2}, \qquad a = 5,6,7,8\,, \qquad t \sim z \sim \psi \sim (\alpha')^0 \tag{B.3}$$

where we have written the four-dimensional base as a GH space (2.5).

The electric charges have also the same scaling as for the black hole:

$$Q_1 \sim Q_2 \sim \alpha', \qquad Q_3 \sim (\alpha')^2 \,. \tag{B.4}$$

Hence, to preserve the fact that the charges of bubbling solutions come entirely from magnetic fluxes, the latter need to scale as

$$k_j^1 \sim k_j^2 \sim \alpha', \qquad k_j^3 \sim (\alpha')^0 \tag{B.5}$$

In particular, we have $r^2 = y_1^2 + y_2^2 + y_3^2$, so $r \sim (\alpha')^2$. At infinity in the M-theory solution the functions Z_I behave like

$$Z_I \sim 1 + \frac{Q_I}{4r} + \ldots \tag{B.6}$$

and so

$$Z_1 \sim \frac{1}{\alpha'} \qquad Z_2 \sim \frac{1}{\alpha'} \qquad Z_3 \sim \text{const}\,. \tag{B.7}$$

So in the limit $\alpha' \to 0$ we can ignore the constant in Z_1 and Z_2 but we should keep it in Z_3. It can be shown that $k \sim A^3 \sim (\alpha')^0$ which finally leads to the desired scaling

$$ds^2_{IIB} \sim \alpha' \,. \tag{B.8}$$

After we have taken the $\alpha' \to 0$ limit we can take the large $r = \frac{\rho^2}{4}$ limit and switch to four-dimensional spherical polar coordinates (4.5), with radial coordinate ρ, in which we have:

$$ds^2_{IIB} \sim \frac{\rho^2}{\sqrt{Q_1 Q_2}}(-dt^2 + dz^2) + \sqrt{Q_1 Q_2}\frac{d\rho^2}{\rho^2} \tag{B.9}$$

$$+\sqrt{Q_1 Q_2}(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2) + \sqrt{\frac{Q_1}{Q_2}} ds^2_{T^4} \tag{B.10}$$

where we have used the freedom to change A^3 by pure gauge transformations. This metric is indeed that of the product space $AdS_3 \times S^3 \times T^4$, where the radius of the AdS_3 and the S^3 is the same and is equal to $(Q_1 Q_2)^{1/4}$. So the bubbling solutions in the decoupling limit are asymptotic to $AdS_3 \times S^3 \times T^4$ and thus should be described by the D1-D5 CFT as expected.[26]

Note that the asymptotic metric in the decoupling limit of *any* of the BPS solutions of section 2 is the same as the metric of the three-charge BPS black hole in the decoupling limit. This implies that the geometries we are analyzing have a field theory description in the same D1-D5 CFT as the three-charge black hole with identical electric charge. The same result was found for supersymmetric black rings [15, 42].

We should also emphasize that in the decoupling limit only the three-charge black holes and the two-charge supertubes have metrics that are everywhere locally $AdS_3 \times S^3 \times T^4$. A general BPS solution like a black ring or a horizonless bubbling solution will have non-trivial geometry and topology.

C The angular momentum of the supertube

Generalities. Our goal in this appendix is to compute the angular momentum of a supertube in the background of three-charge black holes and black rings. Once again we will work in the D0-D4-F1 duality frame:

$$ds^2_{IIA} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt + k)^2 + \sqrt{Z_1 Z_2} ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3} dz^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2)\,. \tag{C.1}$$

For the purpose of our calculations we can restrict without loss of generality to a (non-generic) U(1) × U(1) invariant base metric of the form:

$$ds_4^2 = g_1(u,v)du^2 + g_2(u,v)d\varphi_1^2 + h_1(u,v)dv^2 + h_2(u,v)d\varphi_2^2\,, \tag{C.2}$$

in which the angular momentum vector has the form

$$k = k_1(u,v)d\varphi_1 + k_2(u,v)d\varphi_2\,. \tag{C.3}$$

[26] See [70] for a discussion of a different decoupling limit in which some of these bubbling solutions become dual to microstates of the MSW CFT [54]

The solutions we consider also have RR and NS-NS fields, which have the general form

$$\begin{aligned} B &= (Z_3^{-1}-1)dt\wedge dz + Z_3^{-1}k\wedge dz - B^{(3)}\wedge dz \\ C^{(1)} &= (Z_1^{-1}-1)dt + Z_1^{-1}k - B^{(1)} \\ C^{(3)} &= Z_3^{-1}dt\wedge k\wedge dz - Z_3^{-1}(dt+k)\wedge B^{(1)}\wedge dz + B^{(3)}\wedge dt\wedge dz - f(u,v)d\varphi_1\wedge d\varphi_2\wedge dz \end{aligned} \tag{C.4}$$

where the self-dual harmonic two-forms are $\Theta^{(I)} = dB^{(I)}$, $I = 1,2,3$ and

$$B^{(I)} = B^{(I)}_{\varphi_1}d\varphi_1 + B^{(I)}_{\varphi_2}d\varphi_2\,. \tag{C.5}$$

Consider a probe supertube with world-volume coordinates $\xi = \{\xi^0,\xi^1,\xi^2\equiv\theta\}$ in the above background and suppose that the supertube is embedded as follows:

$$t=\xi^0\,,\qquad z=\xi^1\,,\qquad \varphi_1=\nu_1\theta\,,\qquad \varphi_2=\nu_2\theta \tag{C.6}$$

where $0\le\theta\le 2\pi n_2^{ST}$ and $0\le z\le 2\pi L_z$. The supertube "electric" charges are:

$$\begin{aligned} N_1^{ST} &= \frac{T_{D2}}{T_{D0}}\int dz d\theta \mathcal{F}_{z\theta} = n_2^{ST}\mathcal{F}_{z\theta} \\ N_3^{ST} &= \frac{1}{T_{F1}}\int d\theta\left(\frac{\partial\mathcal{L}_{tot}}{\partial\mathcal{F}_{tz}}\right)\Bigg|_{BPS} = n_2^{ST}\left[Z_2\left(\frac{\nu_1^2 g_2(u,v)+\nu_2^2 h_2(u,v)}{\mathcal{F}_{z\theta}+\nu_1 B^{(3)}_{\varphi_1}+\nu_2 B^{(3)}_{\varphi_2}}\right) - \left(\nu_1 B^{(1)}_{\varphi_1}+\nu_2 B^{(1)}_{\varphi_2}\right)\right] \end{aligned} \tag{C.7}$$

Since the background is independent of φ_1 and φ_2, the supertube has two conserved angular momenta:

$$J^{ST}_{\varphi_1} = \int dz d\theta\frac{\partial\mathcal{L}_{tot}}{\partial\dot\varphi_1}\,,\qquad J^{ST}_{\varphi_2} = \int dz d\theta\frac{\partial\mathcal{L}_{tot}}{\partial\dot\varphi_2}\,. \tag{C.8}$$

One can compute them explicitly and find

$$\begin{aligned} J^{ST}_{\varphi_1} &= n_2^{ST}\Bigg[\nu_1 Z_2 g_2 - \mathcal{F}_{z\theta}B^{(1)}_{\varphi_1} - Z_2 B^{(3)}_{\varphi_1}\left(\frac{\nu_1^2 g_2+\nu_2^2 h_2}{\mathcal{F}_{z\theta}+\nu_1 B^{(3)}_{\varphi_1}+\nu_2 B^{(3)}_{\varphi_2}}\right) \\ &\qquad +\nu_2(B^{(1)}_{\varphi_2}B^{(3)}_{\varphi_1} - B^{(1)}_{\varphi_1}B^{(3)}_{\varphi_2}) + \nu_2 f(u,v)\Bigg]\,, \end{aligned} \tag{C.9}$$

$$\begin{aligned} J^{ST}_{\varphi_2} &= n_2^{ST}\Bigg[\nu_2 Z_2 h_2 - \mathcal{F}_{z\theta}B^{(1)}_{\varphi_2} - Z_2 B^{(3)}_{\varphi_2}\left(\frac{\nu_1^2 g_2+\nu_2^2 h_2}{\mathcal{F}_{z\theta}+\nu_1 B^{(3)}_{\varphi_1}+\nu_2 B^{(3)}_{\varphi_2}}\right) \\ &\qquad +\nu_1(B^{(1)}_{\varphi_1}B^{(3)}_{\varphi_2} - B^{(1)}_{\varphi_2}B^{(3)}_{\varphi_1}) - \nu_1 f(u,v)\Bigg]\,. \end{aligned} \tag{C.10}$$

One can also define a "total" angular momentum of the supertube, as the angular momentum along the direction of the supertube

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} \tag{C.11}$$

and one can show that

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}}\,. \tag{C.12}$$

Flat space. For flat space we have

$$Z_I = 1\,, \qquad B^{(I)}_{\varphi_1} = B^{(I)}_{\varphi_2} = 0\,, \qquad k_1(u,v) = k_2(u,v) = 0\,, \qquad f(u,v) = 0\,, \tag{C.13}$$

and using the change of variables $u = \rho \sin\vartheta$, $v = \rho\cos\vartheta$ one has:

$$g_1(u,v) = h_1(u,v) = 1\,, \qquad g_2 = \rho^2 \sin^2\vartheta\,, \qquad h_2 = \rho^2\cos^2\vartheta\ . \tag{C.14}$$

The conserved "electric" charges of the supertube are

$$N_1^{ST} = n_2^{ST}\mathcal{F}_{z\theta} \tag{C.15}$$

$$N_3^{ST} = n_2^{ST}\left(\frac{\nu_1^2\rho^2\sin^2\vartheta + \nu_2^2\rho^2\cos^2\vartheta}{\mathcal{F}_{z\theta}}\right) \tag{C.16}$$

From these expressions one recovers the familiar radius relation of the supertube

$$\frac{N_1^{ST}N_3^{ST}}{(n_2^{ST})^2} = \rho^2(\nu_1^2\sin^2\vartheta + \nu_2^2\cos^2\vartheta)\ . \tag{C.17}$$

The components of the supertube angular momentum are

$$J^{ST}_{\varphi_1} = \nu_1 n_2^{ST}\rho^2\sin^2\vartheta\,, \tag{C.18}$$

$$J^{ST}_{\varphi_2} = \nu_2 n_2^{ST}\rho^2\cos^2\vartheta\ . \tag{C.19}$$

Of course we again have

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}}\ . \tag{C.20}$$

BMPV black hole. For a BMPV black hole we have

$$Z_I = 1 + \frac{Q_I}{\rho^2}\,, \qquad B^{(I)}_{\varphi_1} = B^{(I)}_{\varphi_2} = 0\,, \qquad k_1 = \frac{J\sin^2\vartheta}{\rho^2}\,, \qquad k_2 = -\frac{J\cos^2\vartheta}{\rho^2}\,, \tag{C.21}$$

$$f = (Z_2 - 1)\rho^2\cos^2\vartheta\,, \qquad g_1(u,v) = h_1(u,v) = 1\,, \tag{C.22}$$

$$g_2 = \rho^2\sin^2\vartheta\,, \qquad h_2 = \rho^2\cos^2\vartheta\,. \tag{C.23}$$

The conserved "electric" charges of the supertube are

$$\begin{aligned} N_1^{ST} &= n_2^{ST}\mathcal{F}_{z\theta}\,,\\ N_3^{ST} &= n_2^{ST}\left(1+\frac{Q_2}{\rho^2}\right)\left(\frac{\nu_1^2\rho^2\sin^2\vartheta + \nu_2^2\rho^2\cos^2\vartheta}{\mathcal{F}_{z\theta}}\right)\ . \end{aligned} \tag{C.24}$$

These again lead to a radius relation for the supertube in the background of the BMPV black hole

$$\frac{N_1^{ST}N_3^{ST}}{(n_2^{ST})^2} = \left(1+\frac{Q_2}{\rho^2}\right)\rho^2(\nu_1^2\sin^2\vartheta + \nu_2^2\cos^2\vartheta)\ . \tag{C.25}$$

The components of the supertube angular momentum are

$$J^{ST}_{\varphi_1} = n_2^{ST}\left[\nu_1\left(1+\frac{Q_2}{\rho^2}\right)\rho^2\sin^2\vartheta + \nu_2 Q_2\cos^2\vartheta\right]\,, \tag{C.26}$$

$$J^{ST}_{\varphi_2} = n_2^{ST}\left[\nu_2\left(1+\frac{Q_2}{\rho^2}\right)\rho^2\cos^2\vartheta - \nu_1 Q_2\cos^2\vartheta\right]\ . \tag{C.27}$$

One can compare this result to the one obtained in [44] where the special case $\nu_1 = n_2^{ST} = 1$, $\nu_2 = 0$ was considered. For these special values (C.26) and (C.27) are identical to (4.4) and (4.5) in [44].

Three-charge BPS black ring. For a three-charge BPS black ring we have:

$$g_1 = \frac{R^2}{(x-y)^2(y^2-1)}\,, \quad g_2 = \frac{R^2(y^2-1)}{(x-y)^2}\,, \quad h_1 = \frac{R^2}{(x-y)^2(1-x^2)}\,, \quad h_2 = \frac{R^2(1-x^2)}{(x-y)^2}\,. \tag{C.28}$$

The functions, Z_I, appearing in the ten-dimensional metric, the one-forms $B^{(I)}$ and the function $f(x,y)$ are given by (4.25), (4.27) and (4.29) respectively. The explicit form of the angular momentum components of the black ring, $k_1(x,y)$ and $k_2(x,y)$, is not needed here.

The conserved "electric" charges of the supertube are

$$N_1^{ST} = n_2^{ST}\mathcal{F}_{z\theta}\,, \tag{C.29}$$

$$\begin{aligned} N_3^{ST} = n_2^{ST}\Bigg[&\frac{n_1}{2}(-\nu_1(d+y)+\nu_2(c+x)) \\ &+\frac{Z_2}{\mathcal{F}_{z\theta}+\frac{n_3}{2}(-\nu_2(c+x)+\nu_1(d+y))}\left(\nu_1^2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2^2R^2\frac{(1-x^2)}{(x-y)^2}\right)\Bigg], \end{aligned} \tag{C.30}$$

which leads to the radius relation

$$\begin{aligned} &\left[N_1^{ST}+\frac{1}{2}n_2^{ST}n_3(\nu_1(y+d)-\nu_2(x+c))\right]\left[N_3^{ST}+\frac{1}{2}n_2^{ST}n_1(\nu_1(y+d)-\nu_2(x+c))\right] = \\ &\qquad (n_2^{ST})^2Z_2\frac{R^2}{(x-y)^2}(\nu_1^2(y^2-1)+\nu_2^2(1-x^2)) \end{aligned} \tag{C.31}$$

The components of the supertube angular momentum are

$$\begin{aligned} J_{\varphi_1}^{ST} = n_2^{ST}\Bigg[&-\mathcal{F}_{z\theta}\frac{n_1}{2}(d+y)+\nu_1Z_2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2 f(x,y) \\ &-Z_2\frac{n_3(d+y)}{2}\left(\frac{\nu_1^2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2^2R^2\frac{(1-x^2)}{(x-y)^2}}{\mathcal{F}_{z\theta}+\frac{n_3}{2}(-\nu_2(c+x)+\nu_1(d+y))}\right)\Bigg] \end{aligned} \tag{C.32}$$

$$\begin{aligned} J_{\varphi_2}^{ST} = n_2^{ST}\Bigg[&\mathcal{F}_{z\theta}\frac{n_1}{2}(c+x)+\nu_2Z_2R^2\frac{(1-x^2)}{(x-y)^2}-\nu_1 f(x,y) \\ &+Z_2\frac{n_3(c+x)}{2}\left(\frac{\nu_1^2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2^2R^2\frac{(1-x^2)}{(x-y)^2}}{\mathcal{F}_{z\theta}+\frac{n_3}{2}(-\nu_2(c+x)+\nu_1(d+y))}\right)\Bigg] \end{aligned} \tag{C.33}$$

And we again have

$$J_{TOT}^{ST} = \nu_1 J_{\varphi_1}^{ST}+\nu_2 J_{\varphi_2}^{ST} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}}\,. \tag{C.34}$$

D Units and conventions

Here we summarize some of the conventions we use in this paper (see [68, 71] for more details).

The tensions of the extended objects in string and M-theory are:

$$T_{F1} = \frac{1}{2\pi\alpha'}\,, \qquad T_{Dp} = \frac{1}{g_s(2\pi)^p(l_s)^{p+1}}\,, \qquad T_{NS5} = \frac{1}{g_s^2(2\pi)^5(l_s)^6}\,, \tag{D.1}$$

$$T_{M2} = \frac{1}{(2\pi)^2(l_{11})^3}\,, \quad T_{M5} = \frac{1}{(2\pi)^5(l_{11})^6} \tag{D.2}$$

where $\alpha' = l_s^2$, l_s is the string length, g_s is the string coupling constant (in the particular duality frame in which one works) and l_D is the D-dimensional Planck length. The Newton's constant in different dimensions is

$$16\pi G_{11} = (2\pi)^8 (l_{11})^9\,, \qquad 16\pi G_{10} = (2\pi)^7 (g_s)^2 (l_s)^8\,, \qquad 16\pi G_D = (2\pi)^{D-3} (l_D)^{D-2}\,. \tag{D.3}$$

One can show that

$$l_{11} = g_s^{1/3} l_s = g_s^{1/3} (\alpha')^{1/2}\,. \tag{D.4}$$

T-duality along a circle of radius R changes the coupling constants to:

$$\widetilde{R} = \frac{\alpha'}{R}\,, \qquad \tilde{g}_s = \frac{l_s}{R} g_s\,, \qquad \tilde{l}_s = l_s\,. \tag{D.5}$$

where $\widetilde{R}$ is the radius after T-duality:

When one compactifies M-theory on a circle of radius L_9, the coupling constants of the resulting type IIA string theory satisfy:

$$L_9 = g_s l_s\,. \tag{D.6}$$

If one compactifies M-theory on a T^6 (along the directions $5, 6, 7, 8, 9, 10$) and the radius of each circle is L_i ($i = \{5, 6, 7, 8, 9, 10\}$), the five-dimensional Newton's constant is

$$G_5 = \frac{G_{11}}{vol(T^6)} = \frac{G_{11}}{(2\pi)^6 L_5 L_6 L_7 L_8 L_9 L_{10}} = \frac{\pi}{4} \frac{(l_{11})^9}{L_5 L_6 L_7 L_8 L_9 L_{10}}\,. \tag{D.7}$$

The relations between the number of M2 and M5 branes, N_I and n_I, and the physical charges of the five-dimensional solution obtained by compactifying M-theory on a T^6, Q_I and q_I, are

$$Q_1 = \frac{(l_{11})^6}{L_7 L_8 L_9 L_{10}} N_1\,, \qquad Q_2 = \frac{(l_{11})^6}{L_5 L_6 L_9 L_{10}} N_2\,, \qquad Q_3 = \frac{(l_{11})^6}{L_5 L_6 L_7 L_8} N_3\,, \tag{D.8}$$

$$q_1 = \frac{(l_{11})^3}{L_5 L_6} n_1\,, \qquad q_2 = \frac{(l_{11})^3}{L_7 L_8} n_2\,, \qquad q_3 = \frac{(l_{11})^3}{L_9 L_{10}} n_3\,. \tag{D.9}$$

We will choose a system of units in which all three T^2 are of equal volume

$$L_5 L_6 = L_7 L_8 = L_9 L_{10} = (l_{11})^3 \equiv g_s l_s^3\,, \tag{D.10}$$

note that this is a numerical identity and is not dimensionally correct since g_s is dimensionless. With this choice we will have

$$G_5 = \frac{\pi}{4}\,, \qquad Q_I = N_I\,, \qquad q_I = n_I\,. \tag{D.11}$$

and these identities hold in every duality frame we use in the paper. Furthermore we will choose

$$g_s l_s = 1\,. \tag{D.12}$$

JHEP07(2009)106

Since we are compactifying M-theory on L_9 we will have $L_9 = g_s l_s = 1$ and $L_{10} = l_s^2$, this implies (note that throughout the paper we put $L_{10} \equiv L_z$)

$$T_{D0} = 1 \,, \qquad 2\pi T_{F1} L_{10} = 1 \,, \qquad \text{and} \qquad \frac{2\pi T_{D2}}{T_{F1}} = 1 \,. \tag{D.13}$$

We have fixed $l_s = g_s^{-1}$ so that a lot of the various brane tension factors, appearing in the probe supertube calculations throughout the paper, cancel. Note that with our choices g_s is still a free parameter but we have fixed the volume of the compactification torii.

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